Question 1 ((1+1+1+1+1+1+1+1+1+1+1+1) + (1+1+1+1+1+1+1+1) = 20 marks)

Start a new answer booklet

a)

i) $P(X=k)=\begin{cases}\dfrac{8!}{k!(8-k)!}0.25^k(0.75)^{8-k} & k\in\{0,1,2,\ldots,8\}\\ 0 & \text{otherwise}\end{cases}$

ii) $E(X)=8(0.25)=2$;

iii) $P(X=3)=\dfrac{8!}{3!5!}(0.25^3)(0.75^5)$;

iv) $P(X>6)=P(X=7)+P(X=8)=8(0.25^7)(0.75)+(0.25^8)$;

v) $P(X\neq 6)=1-\dfrac{8!}{6!2!}(0.25^6)(0.75^2)$;

vi) $P(X=E(X))=P(X=2)=\dfrac{8!}{2!6!}(0.25^2)(0.75^6)$;

vii) $P(Y=E(Y))=P\left(Y=\dfrac{10}{3}\right)=0$;

viii) $P(X>7.3|X>6)=\dfrac{P(X=8)}{P(X=7)+P(X=8)}=\dfrac{0.75^8}{8(0.25)(0.75^7)+0.75^8}$;

ix) $P(X>7.3|X>7)=1$;

x)

$$P\left(\frac{X^2}{Y^2}=9\right)=P(X=3,Y=1)+P(X=6,Y=2)$$
$$=(0.3)\frac{8!}{3!5!}(0.25^3)(0.75^5)+(0.7)(0.3)\frac{8!}{6!2!}(0.25^6)(0.75^2);$$

xi) $P(X=0|XY=0)=1$.

xii) $P(Y>k+n|Y>n)=\dfrac{P(Y>k+n)}{P(Y>n)}=\dfrac{(0.7)^{k+n}}{(0.7)^n}=(0.7)^k=P(Y>k)$

b)

i) $P(A^c)=1-P(A)=0.5$;

ii) $P(B)=P(A\cup B)+P(A\cap B)-P(A)=0.35$;

iii) $P(A|B)=\dfrac{P(A\cap B)}{P(B)}=\dfrac{0.15}{0.35}=\dfrac{3}{7}$

iv) $P(B|A)=\dfrac{P(B\cap A)}{P(A)}=\dfrac{0.15}{0.5}=0.3$;

v) $P(B|A^C\cap B^c)=0$;

vi) $P((A\cup B)^c)=1-P(A\cup B)=0.3$.

vii) $P(A)\times P(B)=0.5\times 0.35\neq 0.15$ hence A and B are not independent.

viii) $P(A\cap C)=0.1$ hence $P(C)=0.2$ and so $P(B\cap C)=0.35\times 0.2=0.07$.

Over…

Do not open your exam paper until instructed.

Question 2 ((2+2+2+2+2+4) + (2+2+2) = 20 marks)

Start a new answer booklet

a) .

i) $g_Y(z) = E(z^Y) = pz + pz(1-p)z + pz(1-p)^2 z^2 + pz(1-p)^3 z^3 + \ldots$.

This is a geometric series, first term pz, common ratio $(1-p)z$. Hence (assuming $-1 < (1-p)z < 1$), $g_Y(z) = \dfrac{pz}{1-(1-p)z}$.

ii) In In order to see the nth success, at least n $Bern(p)$ trials must be observed, hence the range of V is $\{n, n+1, n+2, \ldots\}$

iii) $V \sim NegBin(n,p)$ arises from adding n independent $Geo(p)$ variables. If adding independent random variables, the generating function of the resulting variable can be found by multiplying the generating function of each of the added variables.

This gives, for $V \sim NegBin(n,p)$ $g_V(z) = \left[\dfrac{zp}{1-(1-p)z}\right]^n$.

iv)

$$g_W(z) = \int_0^\infty z^w f(w)dw = \int_0^\infty z^w \lambda e^{-\lambda w} dw =$$

.

$$\lambda \int_0^\infty e^{w\ln(z)} e^{-\lambda w} dw = \lambda \int_0^\infty e^{-w(\lambda - \ln(z))} dw =$$

$$\lambda \left[\frac{e^{-w(\lambda-\ln(z))}}{-(\lambda - \ln(z))}\right]_0^\infty = \frac{\lambda}{\lambda - \ln(z)}.$$

v) $g'_W(z) = \dfrac{\lambda}{z(\lambda - \ln(z))^2}$. Hence $E(W) = g'_W(1) = \dfrac{\lambda}{1(\lambda - \ln(1))^2} = \dfrac{1}{\lambda}$.

vi) For $S = \sum_{i=1}^{Y} W_i$, $g_S(z) = g_Y(g_{W_i}(z))$ hence

$$g_S(z) = g_Y(z) = E(z^Y) = \frac{p\left(\dfrac{\lambda}{\lambda - \ln(z)}\right)}{1-(1-p)\left(\dfrac{\lambda}{\lambda - \ln(z)}\right)} = \frac{p\lambda}{p\lambda - \ln(z)}$$ hence $S \sim \exp(\lambda p)$

Over…

Question 2 (Continued)

b) Let T_A, T_B, T_N respectively be the events that a patient's true status is having disease Type A, disease Type B or neither.

Let D_A, D_B, D_N respectively be the events that a patient's diagnosed status is having disease Type A, disease Type B or neither.

$P(T_A) = 0.01,\ P(T_B) = 0.02,\ P(T_N) = 0.97$.

$P(D_A|T_A) = 0.8,\ P(D_B|T_A) = 0.1,\ P(D_N|T_A) = 0.1$
$P(D_B|T_B) = 0.9,\ P(D_N|T_B) = 0.1$
$P(D_A|T_N) = 0.15,\ P(D_B|T_N) = 0.15,\ P(D_N|T_N) = 0.7$

i) The probability that the selected individual is misclassified by the test is $(0.01)(0.2) + (0.02)(0.1) + (0.97)(0.3) = 0.295$

ii)
$$P(D_N) = P(D_N|T_N)P(T_N) + P(D_N|T_A)P(T_A) + P(D_N|T_B)P(T_B)$$
$$= (0.7)(0.97) + (0.1)(0.01) + (0.1)(0.02) = 0.682$$

iii) $P(T_N|D_N) = \dfrac{(0.7)(0.97)}{0.682} = \dfrac{679}{682}$.

Over…

Question 3 ((1+2+1+2+2+2+2) + (2+1+4+1) = 20 marks)

Start a new answer booklet

a) The probability that:

i) During one whole day, exactly four girls are born is $\frac{e^{-6}6^4}{4!}$.

ii) During one whole day, exactly four babies are born and all are girls is $\frac{e^{-6}6^4}{4!}\frac{e^{-6}6^0}{0!}=\frac{e^{-12}6^4}{4!}$.

iii) During a 12 hour shift, no babies are born from births with complications is $e^{-0.6}$.

iv) From a selection of 10 births selected uniformly at random, all 10 have no complications is 0.9^{10}.

v) During one whole day, at least 11 babies are born given that 6 boys are born without complications and 9 girls are born, of which two were complicated births is 1.

vi) The number of boys born during one hour $\sim Poi(0.25)$

vii) The time (in days) between successive births with complications $\sim \exp(1.2)$.

b)

i) Conditioning on the next move, if the player wins overall from having *k* points, he/she either wins the next point and then wins from having *k*+1 points, or else loses the next point and then wins from having *k*-1 points or draws the next point and then wins from still having *k* points.

Together, these give $W_k=\frac{1}{6}W_{k+1}+\frac{2}{6}W_{k-1}+\frac{3}{6}W_k$ or $3W_k=W_{k+1}+2W_{k-1}$.

ii) $W_0=0$ and $W_{20}=1$.

iii) Seeking a solution of the form $W_k=AM^k$ gives an auxiliary equation of $3M=M^2+2=0$ or $(M-1)(M-2)=0$. The general solution is therefore $W_k=A1^k+B2^k=A+B2^k$.

The boundary conditions tell us that $W_0=0=A+B$ and $W_{20}=1=A+B2^{20}$ so $A=\frac{1}{1-2^{20}}$ and $B=\frac{-1}{1-2^{20}}$ so $W_k=\frac{1-2^k}{1-2^{20}}$.

iv) $W_{15}=\frac{1-2^{15}}{1-2^{20}}\approx 3.12\%$.

Over…

Question 4 ((1+2+1+3+2+1+1+1) + (2+2+2+2)) = 20 marks)

Start a new answer booklet

a) i) $\sum_k P(X=k) = 0.05 + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 1.$

ii) $f(y) = \frac{dF(y)}{dy} = \begin{cases} \frac{\cos(y)}{2} + \frac{1}{\pi} & y \in \left[0, \frac{\pi}{2}\right] \\ 0 & \text{otherwise} \end{cases},$

iii) $E(X) = 0 + \frac{3}{3} + \frac{4}{4} + \frac{5}{5} + \frac{6}{6} = 4;$

iv) $E(Y) = \int_0^{\frac{\pi}{2}} \left(y \frac{\cos(y)}{2} + \frac{y}{\pi} \right) dy = \left[\frac{1}{2} y \sin(y) + \frac{\cos(y)}{2} + \frac{y^2}{2\pi} \right]_0^{\frac{\pi}{2}} = \left(\frac{3\pi}{8} - \frac{1}{2} \right);$

v) $E\left(\frac{1}{X} + X + 3 \right)$ does not take a finite value since $P(X=0) > 0$;

vi) $P(Y=2) = 0$;

vii) $P(X=5 | Y \neq 1) = P(X=5) = 0.2$;

viii) $P(X > Y) = 1 = P(X=0) = 0.95$.

b) Many different possible answers.

Must be a discrete random variable. Total probability must sum to one. Sum of probability between 0 and 2 (inclusive) must be exactly half the sum between 0 and 3 (inclusive.) e.g.

$$P(Q=k) = \begin{cases} 0.5 & k=0 \\ 0.5 & k=3 \\ 0 & \text{otherwise} \end{cases};$$

ii) $P(R=k) = \begin{cases} 1 & k=1.7 \\ 0 & \text{otherwise} \end{cases}.$

iii) Must be a continuous random variable. Total probability must integrate to one. Integral of probability between 0 and 2 (inclusive) must be exactly half the integral between 0 and 3 (inclusive.) e.g.

$$f(s) = \begin{cases} 0.25 & s \in [0,2] \\ 0.5 & s \in (2,3] \\ 0 & \text{otherwise} \end{cases}$$

iv) Must be a continuous random variable. Any function symmetric about zero with large variance will satisfy this. e.g. $T \sim U[-50,50]$ i.e. $f(t) = \begin{cases} 0.01 & t \in [-50,50] \\ 0 & \text{otherwise} \end{cases}$

Over…

Question 5 ((2+2+2+2+2+2) + (3+2+3) = 20 marks)

Start a new answer booklet

a)

i) Taking the columns (and rows) in alphabetical order (i.e. row 1 corresponding to transitions from State A), the transition matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{p_1}{2} & 1-p_1 & 0 & 0 & 0 & \frac{p_1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-p_2 & 0 & 0 & 0 & p_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

ii) If $p_1 = 1, p_2 = 0$ all states are aperiodic i.e. period 1. For example, going from D to D could be done in four moves (DCABD) or three moves (DHED) so there is no periodicity.

iii) If $p_1 = 1, p_2 = 1$ all states have period 2. For example, going from D to D could be done in four moves (DCABD) or six moves (DHIGFED) i.e. always a multiple of 2.

iv) If $p_1 < 1$ all states are aperiodic i.e. period 1. For example, going from D to D could be done in four moves (DCABD) or five moves (DDCABD) so there is no periodicity.

v) Only D can be absorbing. This happens when $p_1 = 0$ (p_2 can take any value.)

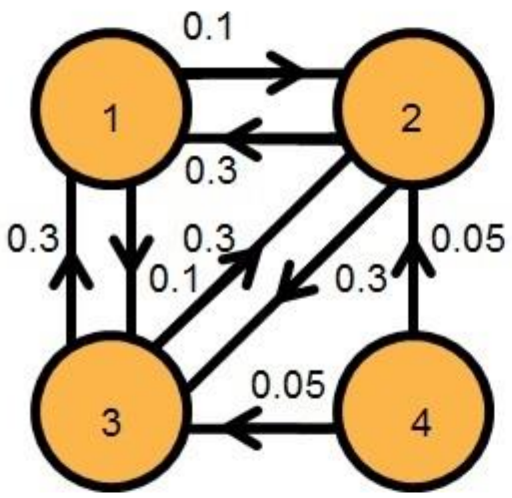
vi) If D is absorbing (i.e. when $p_1 = 0$) all other states are transient.
Also, if $p_2 = 0$ and $p_1 > 0$, only F, G and I are transient.

Over…

Question 5 (Continued)

b)

i)



ii) $P(Y_{n+2} = 2 | Y_n = 3) = (0.3 \times 0.1) + (0.3 \times 0.4) + (0.4 \times 0.3) = 0.27$.

iii) $\Pi_{eq} = (\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4) = (0.6 \quad 0.2 \quad 0.2 \quad 0)$

End of Paper