SEAT NUMBER:

STUDENT NUMBER:

SURNAME:
(FAMILY NAME)

OTHER NAMES:

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They are not to be removed from the exam centre.**

37161_v2 Probability and Random Variables

Time Allowed: 120 minutes.

Reading time: 10 minutes.

Reading time is for reading only. You are not permitted to write, calculate or mark your paper in any way during reading time.

Closed Book
Non-programmable Calculators Only

Permitted materials for this exam:

Drawing Instruments

Materials provided for this exam:

5 x 5 Page Booklet

Students please note:

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- Draft paper (unless provided)
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- Notes (unless specified)

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During the examination **you must first seek permission** (by raising your hand) from a supervisor before:

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Declaration: I declare that I have read the advice above on examination conduct and listened to the examination supervisor's instructions for this exam. In addition, I am aware of the university's rules regarding misconduct during examinations. I am not in possession of, nor do I have access to, any unauthorised material during this examination. I agree to be bound by the university's rules, codes of conduct, and other policies relating to examinations.

Signature:

Date:

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Rough work space

Do **not** write your answers on this page.

Question 1 ((2+2+2+2+2+2+2) + (3+3) = 20 marks)

a) A company employs 40 men and 60 women. At the company's Christmas party, five prizes are given out to employees such that each prize is awarded at random, with each of the 100 employees equally likely to receive each of the prizes.

Assume that the allocation of each prize is independent of that of all other prizes so that it is possible for a single employee to be randomly selected to win more than one prize.

Calculate the probability that:

i) Exactly two of the five prizes are won by women;

ii) The first and second prize drawn are both won by men;

iii) All five prizes go to employees of the same sex (i.e. all are won by men or all are won by women).

iv) Show that the probability that all five prizes are won by different employees $\approx 90.3\%$.

Assume now that after each prize is won, the winning employee's name is removed from consideration for future prizes. That is, the each employee wins the first prize awarded with probability $\frac{1}{100}$ and the second prize is won by each of the employees who did not with the first prize with probability $\frac{1}{99}$ etc.

Calculate the probability that:

v) The first and second prize drawn are both won by men;

vi) All five prizes go to employees of the same sex (i.e. all are won by men or all are won by women).

vii) All five prizes are won by different employees

Over…

Question 1 (Continued)

b) The hiring division of a business examines its records of job applications and their outcomes.
It is noted that 40% of applications are from current employees seeking promotion, 40% of applications are from external candidates who have prior experience in a similar role and 20% of applications come external candidates with no prior experience in a similar role.
40% of candidates applying for promotion were offered the role. 30% of external candidates with prior experience were offered the role and 5% of external candidates with no prior experience were offered the role.
One applicant is selected at random from the list of all applicants, with each person selected with equal probability.

i) Calculate the probability that the selected applicant was offered the role.

ii) Given that the selected candidate was not offered the role, show that the probability that the candidate was an external candidate with no prior experience in a similar role is approximately 27%.

Over…

Question 2 ((2 + 2) + (1+1+1+1+1+1+2+2) + (2+2+2) = 20 marks)

Start a new answer booklet

a) Give an example of a possible random experiment and resulting variable such that the variable would have the following distribution:

i) $R \sim Bern\left(\frac{1}{13}\right)$;   ii) $S \sim Geo\left(\frac{1}{4}\right)$.

For example, rolling a regular fair six-sided die 20 independent times and counting how many times it lands on an even number would give a variable which $\sim Bin\left(20, \frac{1}{2}\right)$.

b) Let A and B be events in a sample space Ω such that

$P(A) = x,\ P(A \cap B) = y$ and $P(A \cup B) = z$ where $0 \leq y \leq x \leq z \leq 1$

Calculate:

i) $P(A^c)$;   ii) $P(B)$;   iii) $P(A|B)$;

iv) $P(B|A)$;   v) $P(B^c | A^C \cap B)$;   vi) $P((A^c \cup B^c))$.

Assume now that the events A and B form a partition of the sample space Ω.

vii) Calculate the value of z.

Assume further that the events A and B form a partition of the sample space Ω and that A and B are independent.

viii) Calculate the value of y.

c) Let T be a continuous random variable with probability density function

$$f(t) = \begin{cases} 0 & t \in [-1,1] \\ \frac{1}{2t^2} & \text{otherwise} \end{cases}.$$

i) Verify that that $f(t)$ is a valid probability density function i.e. $\int_{-\infty}^{\infty} f(t)dt = 1$.

ii) Calculate $E\left(\frac{1}{T}\right)$.

iii) Calculate $P(T > E(T))$.

Over…

Question 3 ((1+1+1+1+1+1+1+1+1+1+1) + (2+1+3+3) = 20 marks)

Start a new answer booklet

a) A hospital's administration office processes patient files 24 hours a day. On average, the office processes 10 files from male patients per hour and 10 files from female patients per hour. 30% of the files relate to child patients and 70% relate to adult patients. The office checks all files for errors while processing. 90% of files are classified as error-free and 10% are classed as containing errors.

Model this scenario by a Poisson process whereby patient files are processed independently of each other. Assume that the gender of each file's patient, the age of each file's patient and whether or not the file contains any errors are all independent.

Calculate the probabilities of the following events:

i) During a one hour period, exactly 15 files are processed;

ii) During a five minute period, no files containing errors are processed;

iii) During a ten minute period, no child files are processed, given no adult files are processed during that ten minutes;

iv) During the first two hours of the day, no files are processed from male adults and no files are processed from female children;

v) The difference between the time that the ninth file is processed and the time that the tenth file is processed is greater than ten minutes;

vi) The first file processed after 1p.m. is for a female patient and contains no errors;

vii) The last three files processed in a given day are all for child patients;

viii) Between 2:00pm and 3:00pm, exactly four files for female children are processed, given exactly three files for female children are processed between 2:00pm and 2:45pm;

ix) At least twenty files are processed in the first hour of the day, given that exactly twenty are processed in the first half hour of the day.

Find the distributions of the following variables:

x) Out of the next ten files processed, the number of files which are adult files containing errors;

xi) The time between the processing of the 60th file and the processing of the 61st file after a given timepoint.

Over…

Question 3 (Continued)

b) A game operates through a system of independent bets. At each turn, a player pays $1. With probability p ($0 < p < 1$) he/she wins $2 (i.e. his/her original stake plus $1 of profit). With probability $q = 1 - p$ he/she loses his/her original stake and wins nothing.

He/she plays the game repeatedly until he/she reaches $0 (“loses”) or reaches $*D* (“wins”), whichever happens first.

Let W_k be the probability that he/she eventually wins given that, at a given time, he/she has $*k*. (Assume *k* and *D* are integers and that $p \neq q$.)

i) Clearly explain why W_k satisfies $W_k = pW_{k+1} + qW_{k-1}$.

ii) Write down the boundary conditions, W_0 and W_D. Explain your answer.

iii) Solve the difference equation in part (i) to show that $W_k = \dfrac{1-\left(\dfrac{q}{p}\right)^k}{1-\left(\dfrac{q}{p}\right)^D}$.

He/she now doubles the amount staked on each bet. That is, with probability *p*, he/she gains $2 profit and with probability $q = 1 - p$ suffers a $2 loss.

Let $\tilde{W}_k$ be the probability that he/she eventually now wins given that, at a given time, he/she has $*k*.

(You may assume that the player’s initial wealth and *D* are the same as in parts i)-iii) and that both are divisible by 2.)

iv) Calculate $\tilde{W}_k$.

Over…

Question 4 ((1+1+1+1+1+1+2+2) + (2+2+4+2) = 20 marks)

Start a new answer booklet

a) Let $Y \sim Bin(10,0.4)$ and $Z \sim Geo(0.3)$ be independent random variables.

Calculate:

i) $E(Y)$; ii) $P(Y=7)$; iii) $P(Z=7)$;

iv) $P(Y=Z=7)$; v) $P(Y=7|Z=7)$; vi) $P(Z>7)$.

vii) In your own words, clearly explain why $Var(E(Y))=0$.

viii) Calculate $E(Var(Y))$. Justify your answer.

b) i) For $W \sim \exp(\lambda)$ show, by differentiating the generating function, that $E(W)=\dfrac{1}{\lambda}$.

ii) For $C \sim Geo(p)$, by differentiating the generating function, calculate $E(C)$.

Let $W_1, W_2, W_3, \ldots$ be independent random variables such that each $W_i \sim \exp(\lambda)$ and let $C \sim Geo(p)$ be independent of each of these.

iii) Show that $V = \sum_{i=1}^{C} W_i$ is an exponential variable and find its rate parameter.

iv) Faults are reported to a 24 hour hotline such that the time (in days) between successive reports are independent variables, each $\sim \exp(20)$.

Each fault is assigned to one of 5 engineering teams, each with probability 0.2, independently of the arrival and assignment of all other faults.

Write down the expected time between successive faults assigned to one team.

Explain your answer in the context of your result for part iii)

Hints: The generating functions of $W \sim \exp(\lambda)$ and $C \sim Geo(p)$ are

$g_W(z) = \dfrac{\lambda}{\lambda - \ln(z)}$ and $g_C(z) = \dfrac{zp}{1-(1-p)z}$ respectively.

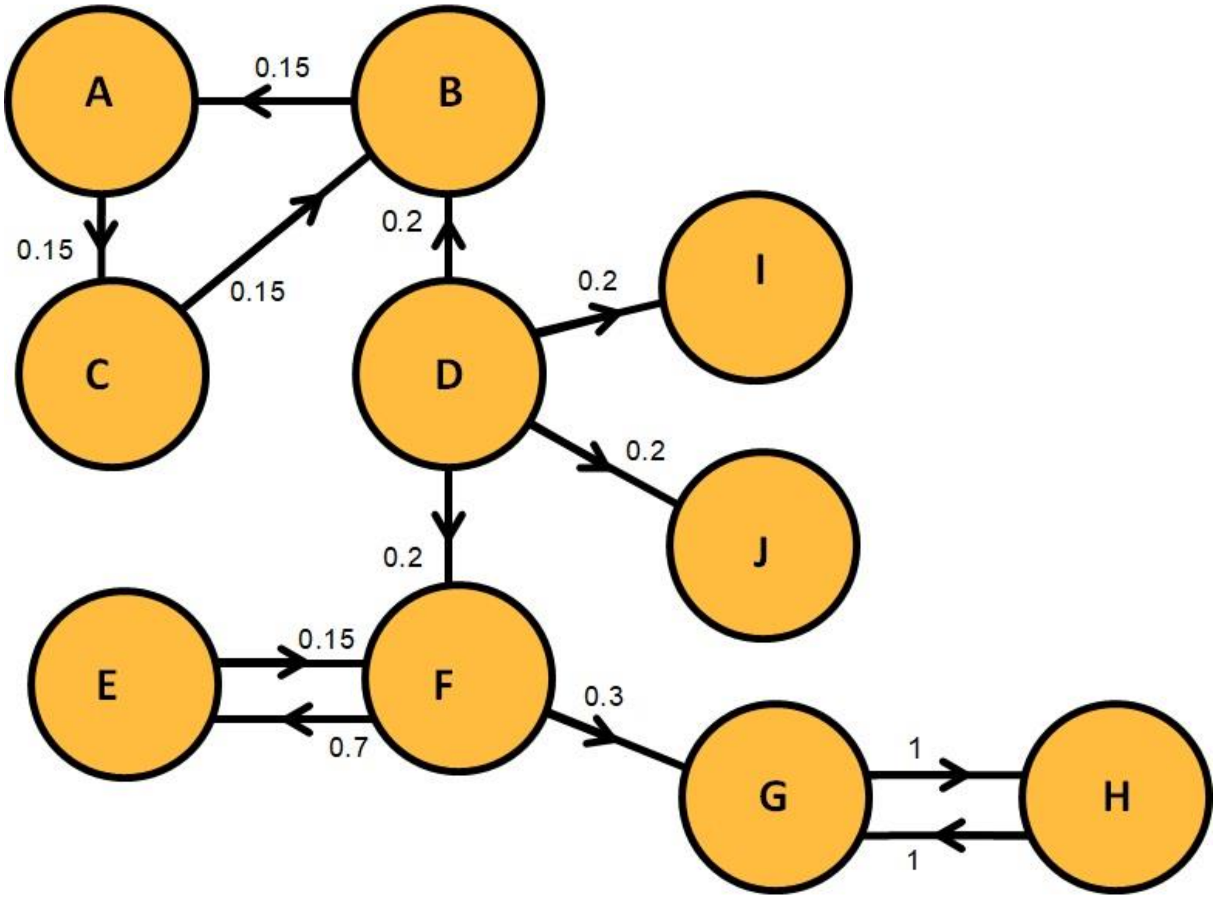
You may use without proof the result that, if $V = \sum_{i=1}^{C} W_i$, $g_V(z) = g_C(g_{W_i}(z))$.

Over…

Question 5 ((2+2+2+2+2) + (2+2+3) + 3 = 20 marks)

Start a new answer booklet

a) Let $X_0, X_1, X_2, ...$ be a Markov Chain which is represented by the state diagram



i) Find the transition matrix for this chain.

An absorbing state is one such that, if the system ever enters that state, the probability that it is ever in a different state is zero.

A persistent state is one such that, if the system is ever in that state and moves to another state, the probability that it never returns is zero.

A transient state is one such that, if the system is ever in that state and moves to another state, the probability that it eventually returns is less than one.

A state *i* is periodic with period $d > 1$ if $P(X_{n+k} = i | X_n = i) = 0$ for all *n* unless *k* is divisible by *d*.

Which of the states for this chain are:

ii) absorbing; iii) persistent; iv) transient?

v) Which states are periodic with period $d > 1$? For each of these states, find the period.

Over…

Question 5 (Continued)

b) Let $Y_0, Y_1, Y_2, \ldots$ be a Markov Chain with transition matrix $P = [p_{ij}] = \begin{pmatrix} 0.25 & 0.5 & 0.25 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.5 & 0.25 \end{pmatrix}$

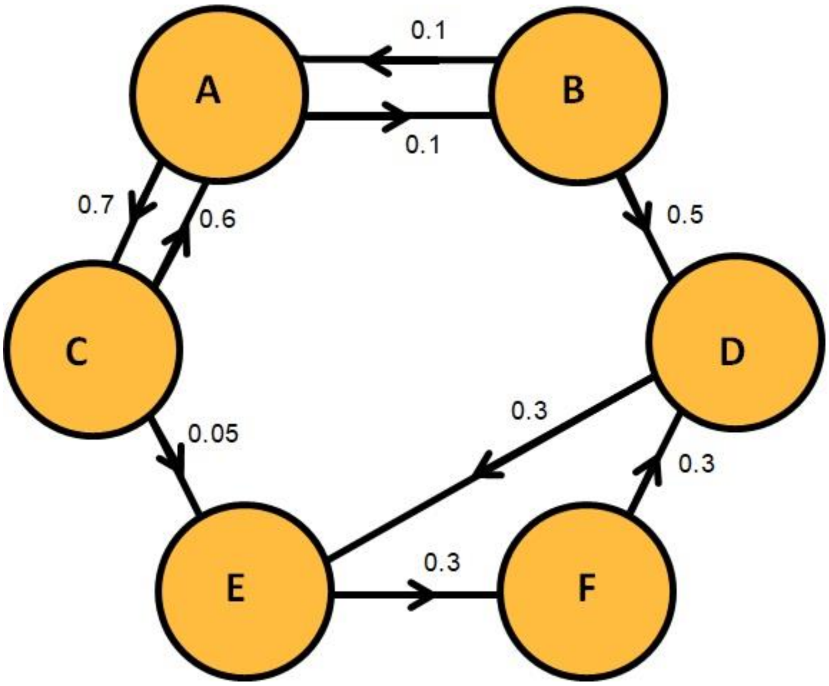
where $p_{ij} = P(Y_{n+1} = j \mid Y_n = i)$

i) Draw the state diagram for this chain.

ii) Calculate the 2-step transition probability $P(Y_{n+2} = 2 \mid Y_n = 3)$.

iii) Calculate the equilibrium distribution for this chain.

c) Let $Y_0, Y_1, Y_2, \ldots$ be a Markov Chain which is represented by the state diagram



By considering the Markov Chain or otherwise, find one eigenvector of .

$$\begin{pmatrix} 0.2 & 0.1 & 0.7 & 0 & 0 & 0 \\ 0.1 & 0.4 & 0 & 0.5 & 0 & 0 \\ 0.6 & 0 & 0.35 & 0 & 0.05 & 0 \\ 0 & 0 & 0 & 0.7 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0.3 \\ 0 & 0 & 0 & 0.3 & 0 & 0.7 \end{pmatrix}$$

State its corresponding eigenvalue

End of Paper

Rough work space

Do **not** write your answers on this page.