Question 1 ((2+2+2+2+2+2+2) + (3+3) = 20 marks)

a) The probability that:

i) Exactly two of the five prizes are won by women is $\binom{5}{2}(0.6)^2(0.4)^3 \approx 0.230$.

ii) The first and second prize drawn both are won by men is $(0.4)^2 = 0.16$.

iii) All five prizes go to employees of the same sex (i.e. all are won by men or all are won by women) is $(0.4)^5 + (0.6)^5 = 0.088$.

iv) All five prizes are won by different employees is $1 \times 0.99 \times 0.98 \times 0.97 \times 0.96 \approx 0.903$.

The probability that:

v) The first and second prize drawn both are won by men is $(0.4) \times \frac{39}{99} = \frac{26}{165}$.

vi) All five prizes go to employees of the same sex (i.e. all are won by men or all are won by women) is $\left[(0.4) \times \frac{39}{99} \times \frac{38}{98} \times \frac{37}{97} \times \frac{36}{96}\right] + \left[(0.6) \times \frac{59}{99} \times \frac{58}{98} \times \frac{57}{97} \times \frac{56}{96}\right]$.

vii) All five prizes are won by different employees is 1.

b) Let O be the event that a candidate is offered the role.

Let C, be the events that a candidate is current employee, X be the event that a candidate is an external candidate with prior experience and N be the event that the candidate is an external candidate with no prior experience.

This gives $P(C) = 0.4$, $P(X) = 0.4$ and $P(N) = 0.2$.

We also have that $P(O|C) = 0.4$, $P(O|X) = 0.3$ and $P(O|N) = 0.05$.

i)
$$\begin{aligned} P(O) &= P(O|C)P(C) + P(O|X)P(X) + P(O|N)P(N) \\ &= 0.4(0.4) + 0.4(0.3) + 0.2(0.05) = 0.29. \end{aligned}$$

ii)
$$P(N|O^c) = \frac{P(N \cap O^c)}{P(O^c)} = \frac{0.2(0.95)}{1-0.29} = \frac{19}{71} \approx 27\%$$

Over…

Do not open your exam paper until instructed.

Question 2 ((2 + 2) + (1+1+1+1+1+1+2+2) + (2+2+2) = 20 marks)

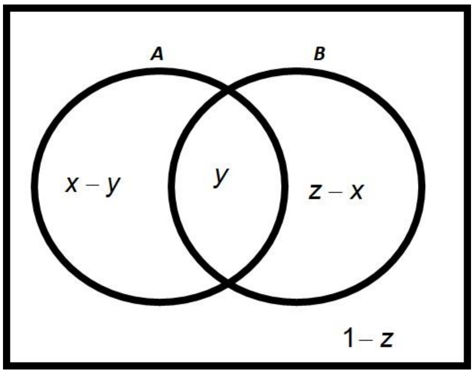
Start a new answer booklet

a) Many possible answers:

i) For example, $R \sim Bern\left(\frac{1}{13}\right)$ could arise from drawing a single card uniformly at random from a regular deck of 52 playing cards and counting how many Kings are selected (either 0 or 1.)

ii) For example, $S \sim Geo\left(\frac{1}{4}\right)$ could arise from drawing cards at uniformly at random from a regular deck of 52 and counting how many cards are drawn until the first Spade is selected, assuming that after each draw, the card is replaced into the deck.

b)



i) $P(A^c) = 1 - x$; ii) $P(B) = z - x + y$;

iii) $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{y}{z - x + y}$; iv) $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{y}{x}$;

v) $P(B^c | A^C \cap B) = 0$; vi) $P((A^c \cup B^c)) = P((A \cap B)^c) = 1 - y$.

vii) If A and B form a partition of the sample space $P(A \cup B) = z = 1$.

viii) If A and B form a partition of the sample space $P(A \cap B) = y = 0$. (The question doesn't ask this, but we further know that, since they are independent, either $P(A) = 0,\ P(B) = 1$ or $P(A) = 1,\ P(B) = 0$

c) .

i) $$\int_{-\infty}^{\infty} f(t)dt = \int_{-\infty}^{-1} \frac{1}{2t^2}dt + \int_{1}^{\infty} \frac{1}{2t^2}dt = \left[\frac{-1}{2t}\right]_{-\infty}^{-1} + \left[\frac{-1}{2t}\right]_{1}^{\infty} = 1.$$

ii) $$E\left(\frac{1}{T}\right) = \int_{-\infty}^{\infty} \frac{1}{t} f(t)dt = \int_{-\infty}^{-1} \frac{1}{2t^3}dt + \int_{1}^{\infty} \frac{1}{2t^3}dt = \left[\frac{-1}{t^2}\right]_{-\infty}^{-1} + \left[\frac{-1}{t^2}\right]_{1}^{\infty} = 0.$$

iii) By symmetry around 0, $P(T > E(T)) - 0.5$.

Over…

Question 3 ((1+1+1+1+1+1+1+1+1+1+1) + (2+1+3+3) = 20 marks)

Start a new answer booklet

a) The probability that:

i) During a one hour period, exactly 15 files are processed: $\dfrac{e^{-20}20^{15}}{15!}$

ii) During a five minute period, no files containing errors are processed: $e^{-1.5}$;;

iii) During a ten minute period, no child files are processed, given no adult files are processed during that ten minutes: e^{-1}.
(Information about adult files is irrelevant.)

iv) During the first two hours of the day, no files are processed from male adults and no files are processed from female children e^{-20};

v) The difference between the time that the ninth file is processed and the time that the tenth file is processed is greater than ten minutes: $e^{-\frac{20}{6}}$;

vi) The first file processed after 1p.m. is for a female patient and contains no errors: $0.5 \times 0.9 = 0.45$.

vii) The last three files processed in a given day are all for child patients: $0.3^3 = 0.027$;

viii) Between 2:00pm and 3:00pm, exactly four files for female children are processed, given exactly three files for female children are processed between 2:00pm and 2:45pm: $\dfrac{0.75 \times e^{-0.75}}{1!}$;

ix) At least twenty files are processed in the first hour of the day, given that exactly twenty are processed in the first half hour of the day: 1
(we are told twenty are processed in the first half hour, hence at least twenty are certainly processed in the first hour.)

The distributions of the following variables:

x) Out of the next ten files processed, the number of files which are adult files containing errors $\sim Bin(10, 0.07)$.

xi) The time between the processing of the 60th file and the processing of the 61st file after a given timepoint $\sim \exp(20)$.

(In hours. Other answers possible in other time units.)

Over…

Question 3 (Continued)

b)

i) Conditioning on the result of the next move, with probability p he/she wins and so his/her chance of winning is equal to that if he/she had started with \$$k$+1. With probability q he/she loses and so his/her chance of winning is equal to that if he/she had started with
\$$k$-1. Hence $W_k = pW_{k+1} + qW_{k-1}$.

ii) $W_0 = 0$ since if he/she ever has no money, he/she cannot win, i.e. his/her chance of winning is 0.
$W_D = 1$ since if he/she ever has \$$D$, he/she has already won with probability 1.

iii) Seeking solutions of $W_k = pW_{k+1} + qW_{k-1}$ of the form $W_k = AM^k$ gives auxiliary equation $pM^2 - M + q = 0 = (pM - q)(M - 1)$ hence $pM^2 - M + q = 0 = (pM - q)(M - 1)$

$W_k = A + B\left(\frac{q}{p}\right)^k$. Initial conditions give $W_k = \dfrac{1 - \left(\frac{q}{p}\right)^k}{1 - \left(\frac{q}{p}\right)^D}$.

iv) Rather than starting k bets away from zero, with the wining boundary D bets away from zero, he/she starts $\frac{k}{2}$ complete bets away from zero and with a boundary $\frac{D}{2}$ bets away, hence $\tilde{W}_k = \dfrac{1 - \left(\frac{q}{p}\right)^{\frac{k}{2}}}{1 - \left(\frac{q}{p}\right)^{\frac{D}{2}}}$.

Over…

Question 4 ((1+1+1+1+1+1+2+2) + (2+2+4+2) = 20 marks)

Start a new answer booklet

a) $E(Y)=4$; ii) $P(Y=7)=\binom{10}{7}0.4^7 0.6^3$; iii) $P(Z=7)=0.3(0.7)^6$;

iv) $P(Y=Z=7)=P(Y=7)P(Z=7)=\binom{10}{7}0.4^7 0.6^3(0.3)(0.7)^6$;

v) $P(Y=7|Z=7)=P(Y=7)=\binom{10}{7}0.4^7 0.6^3$;

vi) $P(Z>7)=0.7^7$.

vii) $E(Y)$ is a constant and hence its variance is zero. $Var(E(Y))=0$.

viii) $Var(Y)$ is a constant and hence its expectation is simply equal to its value. $E(Var(Y))=Var(Y)=10(0.4)(0.6)=2.4$.

b) i) $g_W(z)=\dfrac{\lambda}{\lambda-\ln(z)}$ so $g_W{}'(z)=\dfrac{\lambda\left(\frac{1}{z}\right)}{\left[\lambda-\ln(z)\right]^2}$ so $E(W)=g_W{}'(1)=\dfrac{1}{\lambda}$.

ii) $g_C(z)=\dfrac{zp}{1-(1-p)z}$ so $g_C{}'(z)=\dfrac{\left[1-(1-p)z\right]p+zp(1-p)}{\left[1-(1-p)z\right]^2}$ so $E(C)=g_C{}'(1)=\dfrac{1}{p}$.

iii) $g_V(z)=g_C(g_{W_i}(z))=\dfrac{\dfrac{\lambda}{\lambda-\ln(z)}p}{1-(1-p)\dfrac{\lambda}{\lambda-\ln(z)}}==\dfrac{\lambda p}{\lambda p-\ln(z)}$ so $V=\sum_{i=1}^{C}W_i\sim\exp(\lambda p)$.

iv) If each $W_i\sim\exp(20)$ and $C\sim Geo(0.2)$ then $V=\sum_{i=1}^{C}W_i\sim\exp(4)$ so the expected time between successive faults assigned to one team is $E(V)=0.25$.

$g_W(z)=\dfrac{\lambda}{\lambda-\ln(z)}$ and $g_C(z)=\dfrac{zp}{1-(1-p)z}$ respectively.

You may use without proof the result that, if $V=\sum_{i=1}^{C}W_i$, $g_V(z)=g_C(g_{W_i}(z))$.

Over…

Question 5 ((2+2+2+2+2) + (2+2+3) + 3 = 20 marks)

Start a new answer booklet

a)

i)

$$\begin{pmatrix} 0.85 & 0 & 0.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.15 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.15 & 0.85 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 0.85 & 0.15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

ii) I and J are absorbing; iii) A, B, C, G and H are persistent;

iv) D, E and F are transient.

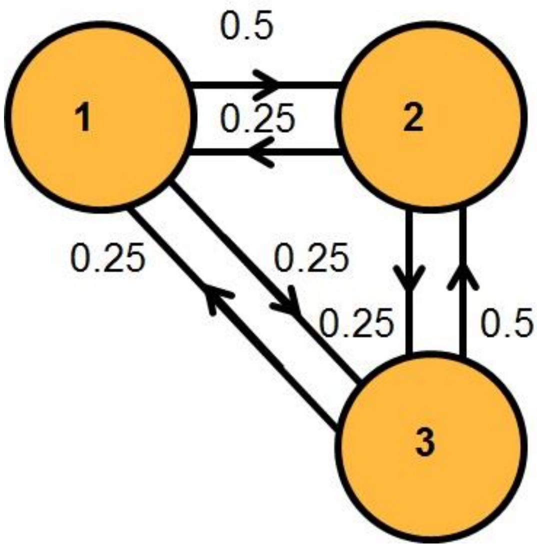
v) Only G and H have period >1. These are both period 2.

Over…

Question 5 (Continued)

b)

i)



ii) $P(Y_{n+2}=2|Y_n=3)=(0.25\times 0.5)+(0.5\times 0.5)+(0.25\times 0.5)=0.5$.

iii) We find $\Pi_{eq}=\begin{pmatrix}A & B & C\end{pmatrix}$ such that $\begin{pmatrix}A & B & C\end{pmatrix}=\begin{pmatrix}A & B & C\end{pmatrix}\begin{pmatrix}0.25 & 0.5 & 0.25\\ 0.25 & 0.5 & 0.25\\ 0.25 & 0.5 & 0.25\end{pmatrix}$ i.e.

$$A=\frac{1}{4}A+\frac{1}{4}B+\frac{1}{4}C,\ \ B=\frac{1}{2}A+\frac{1}{2}B+\frac{1}{2}C,\ \ C=\frac{1}{4}A+\frac{1}{4}B+\frac{1}{4}C.$$

These are solved by any scalar multiple of $\begin{pmatrix}A & B & C\end{pmatrix}=\begin{pmatrix}1 & 2 & 1\end{pmatrix}$ so normalising to get a vector of probabilities gives $\Pi_{eq}=\begin{pmatrix}0.25 & 0.5 & 0.25\end{pmatrix}$.

c) By inspection, we can see that there is an equilibrium distribution which consists of the system being equally likely to be in State D, State E or State F. We therefore can say that the transition matrix must have $\begin{pmatrix}0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}\end{pmatrix}$ as an eigenvector. Its associated eigenvector is 1.

End of Paper

Rough work space
Do **not** write your answers on this page.