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Probability and Random Variables (37161) – Additional Material on Poisson Processes

General Notes:

If we know that we have a Poisson process with independent interarrival times each $\sim \exp(\lambda)$, then we know that the number of arrivals in unit time $\sim Poi(\lambda)$.
Conversely, if we know that the number of arrivals in unit time $\sim Poi(\lambda)$, then we know that the time between successive arrivals is a random variable whose distribution is $\exp(\lambda)$.

Independent Poisson processes can easily be merged simply by adding the rates. If, for example, the number of female customers per hour in a shop $\sim Poi(\lambda_f)$ and the number of male customers $\sim Poi(\lambda_m)$ then the total number of arrival per hour $\sim Poi(\lambda_f + \lambda_m)$.

If arrivals within a Poisson process are split into multiple categories, with no dependence between the order of arrivals and the categories into which they fall then the arrivals within each category form an independent Poisson process. For example if a Poisson process expects 10 arrivals per day and 30% of arrivals are classified as Category 1 and 70% as Category 2, then the number of Category 1 arrivals in a day $\sim Poi(3)$ and the number of Category 2 arrivals in a day $\sim Poi(7)$

The unit of time for a Poisson process can easily be rescaled, with the expected number of arrivals during the new timespan proportionally scaled.
That is, if interarrival times each $\sim \exp(\lambda)$ then the number of arrivals in T time units $\sim Poi(\lambda T)$.
This is intuitive since, for example, if 10 calls are expected in an hour, 50 calls are expected in 5 hours or 2.5 calls are expected in a quarter of an hour etc.

Relative arrivals probabilities are calculated directly through the ratio of arrival rates. For example, if the number of adults arriving in a day $\sim Poi(\lambda_a)$ and the number of children arriving in a day $\sim Poi(\lambda_c)$ then the probability that each arrival is an adult rather than a child $= \dfrac{\lambda_a}{\lambda_a + \lambda_c}$.
Again, this is reasonably intuitive, since if we expect 40 adults per day and 20 children per day, then we expect twice as many adults as children i.e. 2/3 of all arrivals are expected to be adults.

Note that the distributions of interarrival times are dependent upon your definition of unit time. For example, if births occur at an average rate of 5 per hour, then the distribution of the time between successive births $\sim \exp(5)$ if measured in hours. If instead you choose to measure time in days, then the time between successive births $\sim \exp(24 \times 5)$.

In general, if $X \sim Poi(\lambda)$ then $P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$.

Example:

A hotel chain rents out two classes of bedroom, deluxe and standard. On average 80% of bookings are made for standard rooms and 20% for deluxe rooms.
Bookings are taken and paid for in one of three currencies, Australian Dollars, American Dollars and Euros. The times between successive room booking are all modelled by independent exponential variables. It is observed that, on average, a booking is made in Australian Dollars every 15 minutes and a booking is made in American Dollars every 10 minutes. Two bookings per hour are expected from customers paying in Euros.

Assume that the choice of room class is independent of the currency spent.

a) Calculate the probabilities of the following events:

i) Between 10:00am and 10:05am on a given day, no bookings are made.

ii) Between 11:00am and 11:45am, exactly three bookings are made by a customer paying in Euros.

iii) Eleven standard rooms are booked between 1:30pm and 2:30pm given that ten were booked between 1:00pm and 1:30pm.

iv) Eleven standard rooms are booked between 2:30pm and 3:30pm given that ten were booked between 2:30pm and 3:00pm.

v) Between 4:00pm and 4:15pm, only two bookings are made and both are for deluxe rooms.

vi) The next three rooms booked are, successively, a deluxe room booked in American Dollars and then a standard room booked in any currency, then a deluxe room booked in Euros.

b) What is the distribution of the the time between successive bookings made using Australian Dollars.

Notes:

The first step is to get all of the Poisson rates into the same units. If we choose to work in hours, then:
the number of bookings per hour in \$AU $\sim Poi(4)$
the number of bookings per hour in \$US $\sim Poi(6)$
the number of bookings per hour in €EUR $\sim Poi(2)$

These can then be merged and/or split. For example, the total number of bookings per hour $\sim Poi(4+6+2)$.
This gives that the number of bookings made in T hours $\sim Poi(12T)$.

Also, for example, the number of bookings made for deluxe rooms $\sim Poi(12\times 0.2)$.

Solutions:

a)
i) The question is asking about total bookings, so we need to merge the Poisson processes for the three currencies. We also need to rescale the time period to 5 minutes.

We expect 12 bookings per hour, so we expect $12\times\frac{5}{60}$ during each five minute window. This implies bookings during a five minute window $\sim Poi(1)$.

The probability of no bookings made during a five minute window is therefore $\frac{e^{-1}1^0}{0!}=e^{-1}$.

ii) The expected number of customers paying in Euros during a 45 minute period is $2\times\frac{45}{60}$ so the number of Euros bookings during this 45 minute period $\sim Poi(1.5)$.
The probability of exactly three bookings during this period is therefore $\frac{e^{-1.5}(1.5)^3}{3!}$.

iii) The conditioning information is irrelevant. The question is asking about bookings between 1:30pm and 2:30pm so what happened prior to 1:30pm is independent of what follows.

The question is asking for the probability of 11 standard rooms being booked during one hour. The expected number of standard rooms booked per hour is 0.8×12, so the number of standard rooms per hour $\sim Poi(9.6)$.
The probability of 11 standard rooms being booked in an hour is therefore $\frac{e^{-9.6}(9.6)^{11}}{11!}$.

iv) Here, the conditioning information is NOT irrelevant. The question asks about bookings between 2:30pm and 3:30pm and tells us about bookings between 2:30 and 3:00pm.

The period in question i.e. that which we don’t know the number of bookings for is then 30 minutes (between 3:00pm and 3:30pm).

The probability of 11 standard rooms being booked between 2:30pm and 3:30pm, given that ten were booked between 2:30pm and 3:00pm is then simply the probability that one standard room was booked between 3:00pm and 3:30pm.

From the previous part, we know that the number of standard rooms per 30 minutes $\sim Poi(4.8)$ so the probability we seek here is $\frac{e^{-4.8}(4.8)^1}{1!}$.

v)

We are seeking the probability that, during a 15 minute window, two deluxe rooms are booked AND no standard rooms are booked.

The expected number of deluxe rooms booked per 15 minutes is $12 \times 0.2 \times \frac{15}{60}$ and so the number of deluxe rooms booked in this period $\sim Poi(0.6)$.
The expected number of standard rooms booked per 15 minutes is $12 \times 0.8 \times \frac{15}{60}$ and so the number of standard rooms booked in this period $\sim Poi(2.4)$.

The probability that two deluxe rooms and no standard rooms are booked in this period is therefore $\left(\frac{e^{-0.6}(0.6)^2}{2!}\right)\left(\frac{e^{-2.4}(2.4)^0}{0!}\right) = \frac{e^{-3}(0.6)^2}{2!}$.

vi) The next three rooms booked are, successively, a deluxe room booked in American Dollars and then a standard room booked in any currency, then a deluxe room booked in Euros.

Each hour, we expect 12 bookings to be made. Of these 12, we expect $6 \times 0.2 = 1.2$ of these to be deluxe rooms booked in \$US. Thus, the probability that the next room booking is a deluxe room booked in \$US is $\frac{1.2}{12} = \frac{1}{10}$.
Likewise, of the expected 12 bookings per hour, we expect $12 \times 0.8 = 9.6$ of these to be standard room bookings, so the probability that the next room booking is a standard room is $\frac{9.6}{12} = \frac{4}{5}$.
Again, of the expected 12 bookings per hour, we expect $2 \times 0.2 = 0.4$ of these to be deluxe rooms booked in Euros, so the probability that the next room booking in a deluxe room paid for in Euros is $\frac{0.4}{12} = \frac{1}{30}$.
Thus, the probability that next three rooms booked are, successively, a deluxe room booked in American Dollars and then a standard room booked in any currency, then a deluxe room booked in Euros is $\left(\frac{1}{10}\right)\left(\frac{4}{5}\right)\left(\frac{1}{30}\right) = \left(\frac{1}{375}\right)$.

b) The number of bookings taken in \$AU per hour $\sim Poi(4)$ so the time between successive standard room bookings $\sim \exp(4)$.