University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) –
Class 3 Preparation Work
SOLUTIONS

1. i) $$P(N=k)=\begin{cases}(1-p)^2 & k=0\\ 2p(1-p) & k=1\\ p^2 & k=2\\ 0 & \text{otherwise}\end{cases}$$

ii)

$$E(N)=\sum k\times P(N=k)=0\left((1-p)^2\right)+1\left(2p(1-p)\right)+2\left(p^2\right)=2p\,.$$

iii)

$$E(N^2)=\sum k^2\times P(N=k)=0^2\left((1-p)^2\right)+1^2\left(2p(1-p)\right)+2^2\left(p^2\right)=2p+2p^2$$

hence $Var(N)=E(N^2)-E(N)^2=2p+2p^2-(2p)^2=2p(1-p)$

2.

i) Player A wins if and only if:

The first roll is a 5 or a 6 (probability $\frac{1}{3}$) or

The first three rolls are not 5s or 6s, but the fourth is (probability $\frac{1}{3}\left(\frac{2}{3}\right)^3$) or

The first six rolls are not 5s or 6s, but the seventh is (probability $\frac{1}{3}\left(\frac{2}{3}\right)^6$) or… etc.

Hence $P(A)$ is given by the infinite series

$$P(A)=\frac{1}{3}+\frac{1}{3}\left(\frac{2}{3}\right)^3+\frac{1}{3}\left(\frac{2}{3}\right)^6+\frac{1}{3}\left(\frac{2}{3}\right)^9+\frac{1}{3}\left(\frac{2}{3}\right)^{12}+\ldots$$

ii) This is a geometric series, first term $\frac{1}{3}$, common ratio $\left(\frac{2}{3}\right)^3<1$ hence it converges to $P(A)=\dfrac{\frac{1}{3}}{1-\left(\frac{2}{3}\right)^3}=\frac{9}{19}$.

iii) $P(B)=\frac{1}{3}\left(\frac{2}{3}\right)+\frac{1}{3}\left(\frac{2}{3}\right)^4+\frac{1}{3}\left(\frac{2}{3}\right)^7+\frac{1}{3}\left(\frac{2}{3}\right)^{10}+\frac{1}{3}\left(\frac{2}{3}\right)^{13}+\ldots$ and

$$P(C)=\frac{1}{3}\left(\frac{2}{3}\right)^2+\frac{1}{3}\left(\frac{2}{3}\right)^5+\frac{1}{3}\left(\frac{2}{3}\right)^8+\frac{1}{3}\left(\frac{2}{3}\right)^{11}+\frac{1}{3}\left(\frac{2}{3}\right)^{14}+\ldots$$

iv) $P(B)=\left(\frac{2}{3}\right)P(A)$ hence $P(B)=\frac{6}{19}$.

$P(C)=\left(\frac{2}{3}\right)P(B)$ hence $P(B)=\frac{4}{19}$.

Clearly $P(A)+P(B)+P(C)=\frac{9}{19}+\frac{6}{19}+\frac{4}{19}=1$

3.

$$P(X=k)=\begin{cases} \frac{5}{13} & k=0 \\ \frac{1}{13} & k=2 \\ \frac{1}{13} & k=4 \\ \frac{1}{13} & k=6 \\ \frac{1}{13} & k=8 \\ \frac{1}{13} & k=10 \\ \frac{3}{13} & k=-10 \\ 0 & \text{otherwise} \end{cases}$$

i)

$$E(X)=\sum k\times P(X=k)$$

$$=0\left(\frac{5}{13}\right)+2\left(\frac{1}{13}\right)+4\left(\frac{1}{13}\right)+6\left(\frac{1}{13}\right)+8\left(\frac{1}{13}\right)+10\left(\frac{1}{13}\right)-10\left(\frac{3}{13}\right)=0$$

ii) $$P(X^2=k)=\begin{cases} \frac{5}{13} & k=0 \\ \frac{1}{13} & k=4 \\ \frac{1}{13} & k=16 \\ \frac{1}{13} & k=36 \\ \frac{1}{13} & k=64 \\ \frac{4}{13} & k=100 \\ 0 & \text{otherwise} \end{cases}$$

iii)

$$E(X^2) = \sum k^2 \times P(X = k)$$

$$= 0\left(\frac{5}{13}\right) + 4\left(\frac{1}{13}\right) + 16\left(\frac{1}{13}\right) + 36\left(\frac{1}{13}\right) + 64\left(\frac{1}{13}\right) + 100\left(\frac{4}{13}\right) = 40$$

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iv) $Var(X) = E(X^2) - E(X)^2 = 40 - 0^2 = 40$.

4.

a) If $P(A) = 0$, then $P(A \cap B) = P(A)P(B) = 0$ for all *B*, hence *A* and *B* are independent.

Likewise, if $P(A) = 1$, $P(A \cap B) = P(B) = P(A)P(B)$ for all *B*, hence *A* and *B* are independent.

b) $P(A) < P(A|B)$, hence $P(A)P(B) < P(A|B)P(B) = P(A \cap B)$.

$$\frac{P(A)P(B)}{P(A)} < \frac{P(A \cap B)}{P(A)} \text{ hence } P(B) < P(B|A).$$

c)

i) *A* is independent of *A*, hence $P(A \cap A) = P(A)P(A)$. Since $A \cap A = A$,
$P(A) = P(A)^2$ hence $P(A)[P(A) - 1] = 0$, so $P(A) = 00$ or $P(A) = 1$.

ii) The only events which are independent of themselves are those which occur with probability 0 or probability 1. That is, we gain no information by observing the experiment (as we already knew that the event would either certainly happen or certainly not happen.)