University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) –
Class 5 Preparation Work
SOLUTIONS

1. i) $\int_{-\infty}^{\infty} f(x)dx = \int_0^1 kx^4 dx = 1$ Hence $\left[\frac{kx^5}{5}\right]_0^1 = \frac{k}{5} = 1$ so $k = 5$.

ii) $E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 kx^5 dx = \left[\frac{kx^6}{6}\right]_0^1 = \frac{k}{6} = \frac{5}{6}$.

iii)
$$P(0.3 < X < 0.7) = \int_{0.3}^{0.7} kx^4 dx = \left[\frac{kx^5}{5}\right]_{0.3}^{0.7} = \frac{k}{5}\left((0.7)^5 - (0.3)^5\right) = (0.7)^5 - (0.3)^5 .$$

2.

a) $\int_{-\infty}^{\infty} f(y)dy = \int_1^{10} \frac{k}{y} dy = 1$. Hence $\left[k\ln(y)\right]_1^{10} = 1$ so $k = \frac{1}{\ln(10)}$.

b)

i) $P(Y > 2) = \int_2^{10} \frac{k}{y} dy = \left[k\ln(y)\right]_2^{10} = \frac{\ln(10) - \ln(2)}{\ln(10)}$.

ii) $E(Y) = \int_{-\infty}^{\infty} yf(y)dy = \int_1^{10} y\frac{k}{y} dy = \left[ky\right]_1^{10} = 9k = \frac{9}{\ln(10)}$.

iii)
$$Var(Y) = E(Y^2) - E(Y)^2 = \int_{-\infty}^{\infty} y^2 f(y)dy - \left(\frac{9}{\ln(10)}\right)^2$$
$$= \int_1^{10} kydy = \left[\frac{ky^2}{2}\right]_1^{10} - \left(\frac{9}{\ln(10)}\right)^2 = \frac{99}{2\ln(10)} - \frac{81}{(\ln(10))^2} .$$

3.

a) $\int_{-\infty}^{\infty} f(z)dz = \int_{1}^{M} 3dz = 1$ Hence $[3z]_1^M = 3M - 3 = 1$ so $M = \frac{4}{3}$.

b)

i) $E(Z) = \int_{-\infty}^{\infty} zf(z)dz = \int_{1}^{M} 3zdz = \left[\frac{3z^2}{2}\right]_1^M = \frac{3}{2}(M^2 - 1) = \frac{7}{6}$.

ii)
$$Var(Z) = E(Z^2) - E(Z)^2 = \int_{-\infty}^{\infty} z^2 f(z)dz - \left(\frac{7}{6}\right)^2$$
$$= \int_{1}^{M} 3z^2 dz = \left[z^3\right]_1^M - \left(\frac{7}{6}\right)^2 = \frac{37}{27} - \frac{49}{36}.$$

4.

a) $\int_{-\infty}^{\infty} f(t)dt = \int_{-1}^{0} -ktdt + \int_{0}^{3} ktdt = \left[-\frac{kt^2}{2}\right]_{-1}^{0} + \left[\frac{kt^2}{2}\right]_0^3 = 1$.

Hence $5k = 1$ so $k = \frac{1}{5}$.

b) $P(T < 0) = \int_{-1}^{0} -ktdt = \left[-\frac{kt^2}{2}\right]_{-1}^{0} = \frac{k}{2} = 0.1$.

c)

i) $E(T) = \int_{-\infty}^{\infty} tf(t)dt = \int_{-1}^{0} -kt^2 dt + \int_{0}^{3} kt^2 dt = \left[-\frac{kt^3}{3}\right]_{-1}^{0} + \left[\frac{kt^3}{3}\right]_0^3 = \frac{26k}{3} = \frac{26}{15}$

ii)
$$P(T > 2 \mid T > 1) = \frac{P(T > 2 \cap T > 1)}{P(T > 1)} = \frac{P(T > 2)}{P(T > 1)}$$
$$= \frac{\int_2^3 f(t)dt}{\int_1^3 f(t)dt} = \frac{\int_2^3 ktdt}{\int_1^3 ktdt} = \frac{\left[\frac{kt^2}{2}\right]_2^3}{\left[\frac{kt^2}{2}\right]_1^3} = \frac{\left(\frac{5k}{2}\right)}{\left(\frac{8k}{2}\right)} = \frac{5}{8}$$

iii) $P(T > 3 \mid T > 2) = 0$ since $T \leq 3$.