University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) –
Class 6 Preparation Work

1. Consider a random experiment which consists of flipping a (possibly biased) coin such that each flip of the coin lands Heads with probability $0 < p < 1$, independently of all other outcomes.

   a) Let X be the number of flips until the coin first lands Heads. That is, $X \sim Geo(p)$.

      Let T be the number of Heads (0 or 1) shown on the first flip of the coin, so $T \sim Bern(p)$.

      i) Calculate $E(X \mid T = 1)$

      ii) In terms of $E(X)$, calculate $E(X \mid T = 0)$.

      iii) Hence show that $E(E(X \mid T)) = p + (1-p)\left[E(X) + 1\right]$.

      iv) Apply the Law of Total Expectation to show that $E(X) = \dfrac{1}{p}$.

   b) The coin is now flipped n times and Y is defined to be the number of times (out of n) that the coin lands Heads. That is $Y \sim Bin(n, p)$.

      Again, let T be the number of Heads shown on the first flip of the coin.

      i) Write down $Var(Y \mid T = 0)$ and $Var(Y \mid T = 1)$.

      ii) Hence calculate $E(Var(Y \mid T))$.

      iii) In your own words, clearly explain why $E(Y \mid T) = T + (n-1)p$.

      iv) Using parts ii and iii, verify the Law of Total Variance for this problem. That is, show that
      $Var(Y) = E(Var(Y \mid T)) + Var(E(Y \mid T))$.

      (You may assume without proof that $E(Y) = np$ and $\mathrm{Var}(Y) = np(1-p)$.)

2. Let Z be a continuous random variable with density function

$$f(z) = \begin{cases} 3z^2 & z \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Consider the change of variable $V = Z^3$.

Show that V is uniformly distributed and find its range,