University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) –
Class 6 Preparation Work
SOLUTIONS

1.

i) If the first flip is Heads (i.e. $T=1$) then $E(X)=1$ hence $E(X \mid T=1)$.

ii) If the first flip is Tails (i.e. $T=0$) then we have added one flip to the total but are no closer to obtaining the first Heads, hence $E(X \mid T=1)=1+E(X)$.

iii) Conditioning on this first flip, we have that
$E(E(X \mid T))=E(E(X \mid T=1))P(T=1)+E(E(X \mid T=0))P(T=0)$.
As $T \sim Bern(p)$, we have $P(T=1)=p$ and $P(T=0)=1-p$.
Substituting this (along with parts i and ii) gives .
$E(E(X \mid T))=p+(1-p)\left[E(X)+1\right]$

iv) The Law of Total Expectation tells us that $E(E(X \mid T))=E(X)$ hence
$E(X)=p+(1-p)\left[E(X)+1\right]$.
Rearranging gives $E(X)-(1-p)E(X)=1$ hence that $E(X)=\dfrac{1}{p}$.

b) The coin is now flipped n times and Y is defined to be the number of times (out of n) that the coin lands Heads. That is $Y \sim Bin(n,p)$.
Again, let T be the number of Heads shown on the first flip of the coin.

i) If we know the outcome of the first flip, the remaining uncertainty arises from the next (unknown) $n-1$ flips.
Both $Var(Y \mid T=0)$ and $Var(Y \mid T=1)$ therefore equal
$(n-1)p(1-p)$.

ii) $E(Var(Y \mid T))=(n-1)p(1-p)$..

iii) Once we have observed the first outcome (i.e. have knowledge of the value of T), all of our remaining uncertainty about the value of Y comes from the fact that we have $n-1$ additional flips which we have not yet observed. $E(Y \mid T)$ is therefore equal to T

(i.e. 0 or 1) plus however many heads are obtained from the remaining $n-1$ flips.

This is because both are equivalent to observing an additional $n-1$ flips and adding on the first outcome (either 0 or 1.)

iv) Taking the variance of the result from iii gives $Var(E(Y|T)) = Var(T + (n-1)p) = Var(T)$ since $(n-1)p$ is constant and does not add to the variance (the only uncertainty is in the value of *T*.) As $T \sim Bern(p)$, we have that

$Var(E(Y|T)) = Var(T) = p(1-p)$.

Part ii gives that $E(Var(Y|T)) = (n-1)p(1-p)$. Summing these two terms gives

$Var(Y) = E(Var(Y|T)) + Var(E(Y|T)) = (n-1)p(1-p) + p(1-p) = np(1-p)$
as required for $Y \sim Bin(n,p)$.

2. With change of variable $V = Z^3$, we have that $\frac{dv}{dz} = 3z^2 = 3(v^{\frac{2}{3}})$.

We know that the density function of *V* is given by $f(z(v))\frac{dz}{dv}$

$$f(z(v))\frac{dz}{dv} = \left(3v^{\frac{2}{3}}\right)\left(\frac{1}{3v^{\frac{2}{3}}}\right) = 1.$$

As the range of *Z* is $[0,1]$, this is also the range of *V*, since

$V(0) = 0^2 = 0$ and $V(1) = 1^2 = 1$.

This gives $f(v) = \begin{cases} 1 & v \in [0,1] \\ 0 & \text{otherwise} \end{cases}$ i.e. $V \sim U[0,1]$.