University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) –
Class 8 Preparation Work

1. A card is drawn from a standard deck (of 52) playing cards with all cards equally likely to be selected.

   The random variable X takes the value 5 if the card is a picture card (Jack, Queen or King), the value 3 if the card is an Ace and the value -2 if the card is a 2,3,4,5,6,7,8,9 or 10.

   i) Write down the probability mass function. That is, for all values of *k,* write down $P(X=k)$.

   ii) Hence find $E(z^X)$, the generating function of X.

   iii) Show that the expected value of X is 0.

2.

   If $X \sim Bin(N,p)$ then $g_X(z)=E(z^X)=\left[(1-p)+pz\right]^N$.

   a) Given that a binomial random variable arises as the sum of independent identical Bernoulli variables, each with the same probability parameter, or otherwise, find the generating function of $T \sim Bern(p)$.

   b) Let $T_1,T_2,...,T_X$ each be independent variables such that each $T_i \sim Bern(p)$ and where $X \sim Bin(N,p)$. Let $Y=\sum_{i=0}^{X} T_i$. Show that Y is a binomial random variable and find both its parameters.

      Hint: You may use without proof the result that $g_Y(z)=g_X(g_{T_i}(z))$.

   c) A player is given 45 fair (six-sided) dice. She initially rolls each once. For each die, if the result is a 1 or a 6, the player keeps the die. If the result is a 2,3,4 or 5, the player loses the die.

      After this initial round, she re-rolls all her <u>remaining</u> dice and declares each to be a winning die if its result is a 3 or a 4.

      i) Explain why the expected number of winning dice is 5.

      ii) Find the variance of the number of winning dice.

3. A geometric random variable $X \sim Geo(p)$ can be obtained by counting the number of independent Bernoulli $Bern(p)$ trials until the first "success" or 1. That is, $P(X=k)=\begin{cases} p(1-p)^{k-1} & k \in \{1,2,3,...\} \\ 0 & \text{otherwise} \end{cases}$.

a) i) Find the generating function of X, $g_X(z)=E(z^X)=\sum_{k=1}^{\infty} z^k P(X=k)$.

ii) By differentiating the generating function, find $E(X)$.

An exponential random variable $Y \sim \exp(p)$ has density function

$$f(y)=\begin{cases} \lambda e^{-\lambda y} & y \in [0,\infty) \\ 0 & \text{otherwise} \end{cases}.$$

b) i) Show that the generating function of Y, $g_Y(z)=E(z^Y)=\dfrac{\lambda}{\lambda-\ln(z)}$.

ii) Buses arrive at a bus stop independently of each other, such that the time between successive arrivals (in minutes) $\sim \exp(0.1)$.

Write down the expected time between successive bus arrivals (in minutes.)

c) A traveller decides to play a game. He repeatedly picks a card from a standard deck of 52 playing cards. He counts how many cards he needs to draw (replacing each card into the deck and reshuffling after each election) until he first selects a Spade. He then waits until that number of buses have arrives before boarding one.

That is, if the first card he draws is a Spade, he boards the first bus. If he does not draw a Spade until the second card, he boards the second bus etc.

What is the distribution of how long he will wait before boarding a bus?

Hint: You may use without proof the result that, if $W=\sum_{i=1}^{X} Y_i$, $g_W(z)=g_X(g_{Y_i}(z))$.