University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) –
Class 8 Preparation Work
SOLUTIONS

1.

i) $$P(X=k)=\begin{cases}\dfrac{3}{13} & k=5\\ \dfrac{1}{13} & k=3\\ \dfrac{9}{13} & k=-2\\ 0 & \text{otherwise}\end{cases}.$$

ii) $$E(z^X)=g_X(z)=\sum_k z^k P(X=k)=\frac{3}{13}z^5+\frac{1}{13}z^3+\frac{9}{13}z^{-2},$$

iii) $$E(X)=g_X'(1)=\left[\frac{15}{13}z^2+\frac{3}{13}z^2+\frac{-18}{13}z^{-3}\right]_{z=1}=0$$

2.

a) A Bernoulli variable is equivalent to a binomial variable for just one trial (with the same probability parameter) hence the generating function of $T \sim Bern(p)$ is $g_T(z) = E(z^T) = \left[(1-p)+pz\right]^1 = 1-p+pz$.

b) We know that $g_Y(z) = g_X(g_{T_i}(z))$.

Since $X \sim Bin(N,p)$ then $g_X(z) = E(z^X) = \left[(1-p)+pz\right]^N$.
Likewise, each $T_i \sim Bern(p)$ so $g_{T_i}(z) = 1-p+pz$.

Putting these together, we find $g_Y(z) = g_X(g_{T_i}(z)) = \left[(1-p)+p\left[1-p+pz\right]\right]^N$.

This gives $g_Y(z) = \left[1-p^2+p^2z\right]^N$ hence $Y \sim Bin(N,p^2)$.

c) i) The number of dice which are kept after one roll $\sim Bin\left(45,\frac{1}{3}\right)$. Each die which is rolled a second time becomes a winner with probability $\frac{1}{3}$, each independently of all others. Hence, the number of winners is the sum of $T_1,T_2,...,T_X$ where each $T_i \sim Bern\left(\frac{1}{3}\right)$ and $X \sim Bin\left(45,\frac{1}{3}\right)$. We know from the previous part that this means the number of winners $\sim Bin\left(45,\frac{1}{9}\right)$. This gives an expected number of winners $= 45 \times \frac{1}{9} = 5$.

(This is to be expected, since each die (independently) is rolled for a second time with probability $\frac{1}{3}$ and each of these is a winner with probability $\frac{1}{3}$. Hence, we expect $\frac{1}{9}$ of the 45 dice to be winners.)

ii) The variance of a $Bin\left(45,\frac{1}{9}\right)$ variable is $45 \times \frac{1}{9} \times \left(1-\frac{1}{9}\right) = \frac{40}{9}$.

3.

a) i)

$$g_X(z) = E(z^X) = \sum_{k=0}^{\infty} z^k p(1-p)^{k-1} = pz + pz(1-p)z + pz(1-p)^2 z^2 + pz(1-p)^3 z^3 + \ldots$$

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This is a geometric series, first term pz, common ratio $(1-p)z$. Hence (assuming $-1 < (1-p)z < 1$) $g_X(z) = \dfrac{pz}{1-(1-p)z}$.

ii) Differentiating this gives

$$g'_X(z) = \left[\frac{pz}{1-(1-p)z}\right]' = \frac{p\left[1-(1-p)z\right] - pz\left[-(1-p)\right]}{\left[1-(1-p)z\right]^2}$$ hence

$$E(X) = g'_X(1) = \frac{1}{p}.$$

b)

i)

$$g_Y(z) = E(z^Y) = \int_0^{\infty} \lambda e^{-\lambda y} z^y dy = \lambda \int_0^{\infty} e^{-\lambda y} e^{y \log(z)} dy = \int_0^{\infty} e^{-y(\lambda - \log(z))} dy = \frac{\lambda}{\lambda - \log(z)}.$$

ii) The expectation of $Y_i \sim \exp(0.1)$ is $E(Y_i) = \dfrac{1}{0.1} = 10$.

c) We know $g_X(z) = \dfrac{pz}{1-(1-p)z}$ and $g_{Y_i}(z) = \dfrac{\lambda}{\lambda - \log(z)}$ so if

$$W = \sum_{i=0}^{X} Y_i\,,\quad g_W(z) = g_X(g_{Y_i}(z)) =$$

$$g_X(z) = \frac{p\dfrac{\lambda}{\lambda-\log(z)}}{1-(1-p)\dfrac{\lambda}{\lambda-\log(z)}} = \frac{p\lambda}{\lambda - \log(z) - \lambda + p\lambda} = \frac{p\lambda}{p\lambda - \log(z)}.$$

That is, if $X \sim Geo(p)$ and each $Y_i \sim \exp(\lambda)$ then $W = \sum_{i=0}^{X} Y_i \sim \exp(\lambda p)$.

In this case, the number of cards drawn until the first Spade $\sim Geo(0.25)$ and the time between each bus $\sim \exp(0.1)$ so the time he waits $\sim \exp(0.025)$.