University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) – Tutorial/Laboratory 10

1. A car is parked each evening in (exactly) one of four parking spaces. The position of the car each evening is modelled with a Markov Chain such that, if the car is in Space A one day, there is a 70% chance it will be in Space A the following day and a 10% chance it will be in each of the other three spaces. If the car is in Space B or Space C one day, then there is a 25% chance of it being in each of the four spaces the following day an if the car is in Space D one day, then there is a 40% chance it will be in Space A, a 30% chance of it being in State B and a 30% chance of it being in State C one day later.

   i) Write down the transition matrix for this Markov Chain (with the rows corresponding to transitions from states A, B, C and D in that order.)

   ii) If the car is in Space A on Monday evening, calculate the probability that it is also in Space A on Wednesday evening.

   iii) If the car is in Space A on Monday evening, calculate the probability that, between Monday evening and Thursday evening, the car is parked once in each of the four spaces. Show all of your working.

   iv) Calculate the equilibrium distribution for this Markov Chain, Show all of your working.

2. Let $X_0, X_1, X_2, ...$ be a Markov Chain with transition matrix

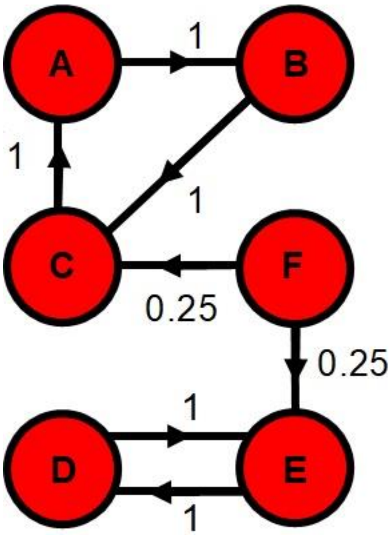
$$P = [p_{ij}] = \begin{pmatrix} 0.4 & 0.3 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where $p_{ij} = P(X_{n+1} = j \mid X_n = i)$

i) Draw the state diagram for this chain.

ii) Write down all of the (linearly distinct) equilibrium distributions for this chain. Justify your answers.

3. Consider the Markov Chain $Y_0, Y_1, Y_2, ...$ with state diagram



i) Write down the transition matrix for this Markov Chain (with the rows corresponding to transitions from states A, B, C, D, E and F in that order.)

ii) Which is the only possible starting state for which the eventual equilibrium distribution reaches is not certain? Justify your answer.

iii) Calculate $P\left(Y_{n+6} = (D \cup A) \mid Y_n = (D \cup A)\right)$ for any $n \in \mathbb{Z}^+$.
Justify your answer.

4. A player plays a board game whereby he/she starts on Square 12 and rolls a regular fair die repeatedly. Each time the die shows a 6, the player moves to a square one number higher (e.g. from Square 3 to Square 4). Each time the die shows a 1 or a 2, the player moves to a square one number lower (e.g. from Square 7 to Square 6). Each time the player rolls a 3, 4 or a 5, the player does not move.

The game ends when the player first reaches Square 1 (“loses”) or Square 15 (“wins”).

Let W_k be the probability that the player wins the game, given that he/she is on Square $k \in \{1,2,3,...,15\}$.

i) Showing all of your working, derive a second order difference equation which must be satisfied by W_k

ii) Find the general solution of the difference equation. Your solution should include two undetermined constants.

iii) Write down the boundary conditions. Justify your answer.

iv) Hence show that, from the player’s starting position, the probability that he/she wins the game is approximately $\frac{1}{8}$.

Now, the player wishes to know how many more times he/she expects to roll the die before the game ends, given his/her current position.

Let C_k be the expected number of additional rolls until the game ends, given that he/she is on Square $k \in \{1,2,3,...,15\}$.

v) Showing all of your working, derive a second order difference equation which must be satisfied by C_k

vi) Write down the boundary conditions. Justify your answer.

vii) Given that the general solution of the difference equation is $C_k = A + B2^k + 6k$, calculate which value of k maximises the expected number of additional games. Justify your answer.