University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) – Tutorial/Laboratory 10
TO BE HANDED IN FOR ASSESSMENT

Please hand in your answers, showing all relevant working in the spaces provided. You may use additional sheets for rough working, but only these worksheets should be submitted for assessment.

1. A car is parked each evening in (exactly) one of four parking spaces. The position of the car each evening is modelled with a Markov Chain such that, if the car is in Space A one day, there is a 70% chance it will be in Space A the following day and a 10% chance it will be in each of the other three spaces. If the car is in Space B or Space C one day, then there is a 25% chance of it being in each of the four spaces the following day an if the car is in Space D one day, then there is a 40% chance it will be in Space A, a 30% chance of it being in State B and a 30% chance of it being in State C one day later.

   i) Write down the transition matrix for this Markov Chain (with the rows corresponding to transitions from states A, B, C and D in that order.)

$$\begin{pmatrix} 0.7 & 0.1 & 0.1 & 0.1 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.4 & 0.3 & 0.3 & 0 \end{pmatrix}$$

   ii) If the car is in Space A on Monday evening, calculate the probability that it is also in Space A on Wednesday evening.

Summing the probabilities of all four ways this can happen (e.g. A then A then A or A then B then A etc.) we obtain

$$(0.7)(0.7)+(0.1)(0.25)+(0.1)(0.25)+(0.1)(0.4)=0.58$$

iii) If the car is in Space A on Monday evening, calculate the probability that, between Monday evening and Thursday evening, the car is parked once in each of the four spaces. Show all of your working.

There are six ways for this to happen e.g. ABCD. ABDC, ACBD, ACDB, ADBC, ADCB.

Summing these probabilities gives

$$
\begin{aligned}
&(0.1)(0.25)(0.25)+(0.1)(0.25)(0.3)\\
&+(0.1)(0.25)(0.25)+(0.1)(0.25)(0.3)\\
&+(0.1)(0.3)(0.25)++(0.1)(0.3)(0.25)\\
&=0.0425
\end{aligned}
$$

iv) Calculate the equilibrium distribution for this Markov Chain, Show all of your working.

We solve

$$
\begin{pmatrix} A & B & C & D \end{pmatrix}\begin{pmatrix} 0.7 & 0.1 & 0.1 & 0.1 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.4 & 0.3 & 0.3 & 0 \end{pmatrix} = \begin{pmatrix} A & B & C & D \end{pmatrix}
$$

$$
\begin{aligned}
0.7A+0.25B+0.25C+0.4D&=A\\
0.1A+0.25B+0.25C+0.3D&=B\\
0.1A+0.25B+0.25C+0.3D&=C\\
0.1A+0.25B+0.25C\qquad&=D
\end{aligned}
$$

We can see that $B=C$ and $\begin{aligned}0.5B+0.4D&=0.3A\\ 0.1A+0.5B&=D\end{aligned}$ hence $(10D-A)+4D=3A$.

We can solve this with $14D=4A$, so we have $\begin{pmatrix} A & B & C & D \end{pmatrix}=\begin{pmatrix} 14 & \frac{26}{5} & \frac{26}{5} & 4 \end{pmatrix}$.

Normalising to obtain a probability vector, we obtain

$$
\frac{5}{142}\begin{pmatrix} 14 & \frac{26}{5} & \frac{26}{5} & 4 \end{pmatrix}=\begin{pmatrix} \frac{35}{71} & \frac{13}{71} & \frac{13}{71} & \frac{10}{71} \end{pmatrix}.
$$

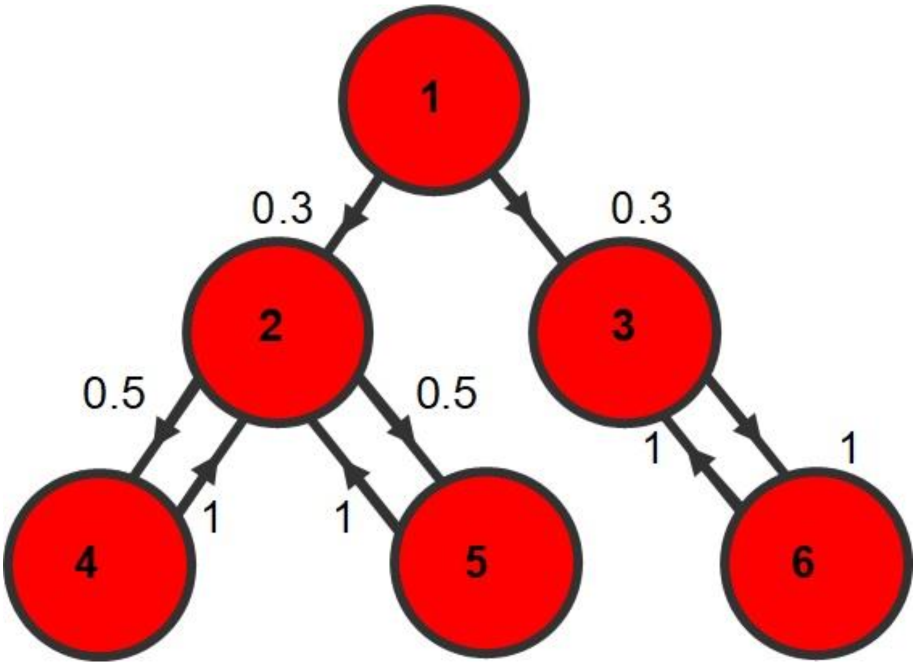
(10 marks)

2. Let $X_0, X_1, X_2, \ldots$ be a Markov Chain with transition matrix

$$P = [p_{ij}] = \begin{pmatrix} 0.4 & 0.3 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where $p_{ij} = P(X_{n+1} = j \mid X_n = i)$

i) Draw the state diagram for this chain.



ii) Write down all of the (linearly distinct) equilibrium distributions for this chain. Justify your answers.

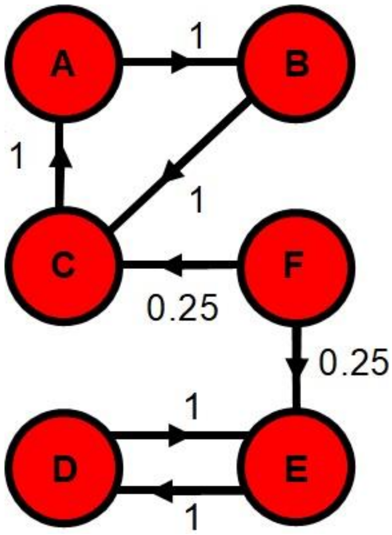
There are two distinct equilibria: Either oscillating between 3 and 6 with both equally likely or else alternating between 2 and either 4 or 5 such that every second move is into 2 and the others are equally split between 4 and 5.

$\Pi_{eq} = \begin{pmatrix} 0 & 0 & 0.5 & 0 & 0 & 0.5 \end{pmatrix}$ or

$\Pi_{eq} = \begin{pmatrix} 0 & 0.5 & 0 & 0.25 & 0.25 & 0 \end{pmatrix}$

(5 marks)

3. Consider the Markov Chain $Y_0, Y_1, Y_2, \ldots$ with state diagram



i) Write down the transition matrix for this Markov Chain (with the rows corresponding to transitions from states A, B, C, D, E and F in that order.)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.25 & 0 & 0.25 & 0.5 \end{pmatrix}$$

ii) Which is the only state for which the eventual equilibrium distribution reaches is not certain? Justify your answer.

Only F has any uncertainty about which eventual equilibrium distribution it will end in. It communicates with all states whereas D and E only communicate with each other, as to A, B and C.

iii) Calculate $P\left(Y_{n+6} = (D \cup A) \mid Y_n = (D \cup A)\right)$ for any $n \in \mathbb{Z}^+$.
Justify your answer.

If the chain is in state D or A, then it is certainly in D or A six steps in the future since D has period 2 and A had period 3.

$$P\left(Y_{n+6} = (D \cup A) \mid Y_n = (D \cup A)\right) = 1$$

(5 marks)

4. A player plays a board game whereby he/she starts on Square 12 and rolls a regular fair die repeatedly. Each time the die shows a 6, the player moves to a square one number higher (e.g. from Square 3 to Square 4). Each time the die shows a 1 or a 2, the player moves to a square one number lower (e.g. from Square 7 to Square 6). Each time the player rolls a 3, 4 or a 5, the player does not move.

The game ends when the player first reaches Square 1 (“loses”) or Square 15 (“wins”).

Let W_k be the probability that the player wins the game, given that he/she is on Square $k \in \{1,2,3,...,15\}$.

i) Showing all of your working, derive a second order difference equation which must be satisfied by W_k

Conditioning on the next outcome, we have

$$W_k = P(\text{Win next move})W_{k+1} + P(\text{Lose next move})W_{k-1} + P(\text{Draw next move})W_k$$

$W_k = \frac{1}{6}W_{k+1} + \frac{2}{6}W_{k-1} + \frac{3}{6}W_k$ hence $3W_k = W_{k+1} + 2W_{k-1}$.

ii) Find the general solution of the difference equation. Your solution should include two undetermined constants.

To solve $3W_k = W_{k+1} + 2W_{k-1}$, we seek a solution of the form $W_k = AM^k$.

This gives $W_{k+1} = AM^{k+1} = MW_k$ etc hence

$M^2 - 3M + 1 = (M-1)(M-2) = 0$.

The general solution is then $W_k = A_1 1^k + A_2 2^k$ for undetermined constants A_1 and A_2..

iii) Write down the boundary conditions. Justify your answer.

$W_1 = 0$ (if the player reaches 1, he/she has certainly not won.)

$W_{15} = 1$ (if the player reaches 15, he/she has certainly won.)

iv) Hence show that, from the player's starting position, the probability that he/she wins the game is approximately $\frac{1}{8}$.

$W_1 = A_1 + 2A_2 = 0$ and $W_{15} = A_1 + 2^{15} A_2 = 1$ hence

$(-2A_2) + 2^{15} A_2 = 1$ so $A_2 = \frac{-1}{2(1-2^{14})}$ and $A_1 = \frac{1}{(1-2^{14})}$.

This gives $W_k = \frac{1-2^{k-1}}{1-2^{14}}$ and, specifically,

$$W_{12} = \frac{1-2^{11}}{1-2^{14}} \approx \frac{2^{11}}{2^{14}} = \frac{1}{8}$$

Now, the player wishes to know how many more times he/she expects to roll the die before the game ends, given his/her current position.

Let C_k be the expected number of additional rolls until the game ends, given that he/she is on Square $k \in \{1,2,3,...,15\}$.

v) Showing all of your working, derive a second order difference equation which must be satisfied by C_k

Conditioning on the next outcome, we have

$$\begin{aligned} C_k = &P(\text{Win next move})\left[C_{k+1} + 1\right] \\ &+P(\text{Lose next move})\left[C_{k-1} + 1\right] \\ &+P(\text{Draw next move})\left[C_k + 1\right] \end{aligned}$$

(since the counter of games played increases by one, regardless of the result.)

$C_k = \frac{1}{6}C_{k+1} + \frac{2}{6}C_{k-1} + \frac{3}{6}C_k + 1$ hence $3C_k = C_{k+1} + 2C_{k-1} + 6$.

vi) Write down the boundary conditions. Justify your answer.

$C_1 = C_{15} = 0$ since if the player ever reaches 1 or 15, no more games are required.

vii) Given that the general solution of the difference equation is $C_k = A + B2^k + 6k$, calculate which value of k maximises the expected number of additional games. Justify your answer.

$C_k = A + B2^k + 6k$. The boundary conditions tell us that $C_1 = A + 2B + 6 = 0$ and $C_1 = A + 2^{15}B + 90 = 0$.

$(2^{15} - 2)B + 84 = 0$ hence $B = \frac{42}{1-2^{14}}$ and $A = -6 - \frac{84}{1-2^{14}}$.

Enumerating various terms, we see that

$C_{11} \approx 54.75$
$C_{12} \approx 55.50$
$C_{13} \approx 51.00$

The function only has one turning point between 1 and 15, hence the expected number of games is maximised when starting in Square 12.

(12 marks)

5. Written task. Please see the marking rubric on Canvas before attempting this problem.

In your own words, clearly define a Markov Chain.

Give an example of two sequences of random variables, $X_1, X_2, X_3, ...$ and $Y_1, Y_2, Y_3, ...$ both of which take values from the set of states $\{1, 2, 3, 4, 5, 6\}$ such that $X_1, X_2, X_3, ...$ is not a Markov Chain but $Y_1, Y_2, Y_3, ...$ is a Markov Chain.

Define your variables in everyday language, not mathematical notation

A Markov Chain is a sequence of random variables such that the conditional probability of a future observation, given the present observation is exactly the same as the conditional probability of that observation given the present and past values.

Let $Y_1, Y_2, Y_3, ...$ be the outcomes of a sequence of independent rolls of a regular fair six-sided die. This clearly forms a Markov Chain as no additional information is gained by knowing past outcomes.

Let $X_1, X_2, X_3, ...$ be the value of the largest value obtained on the previous two rolls of a sequence of independent rolls of a regular fair six-sided die. This clearly does not form a Markov Chain as our observations also depend on past values.

(3 marks, including assessment for English Language)