University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) – Tutorial/Laboratory 11
TO BE HANDED IN FOR ASSESSMENT

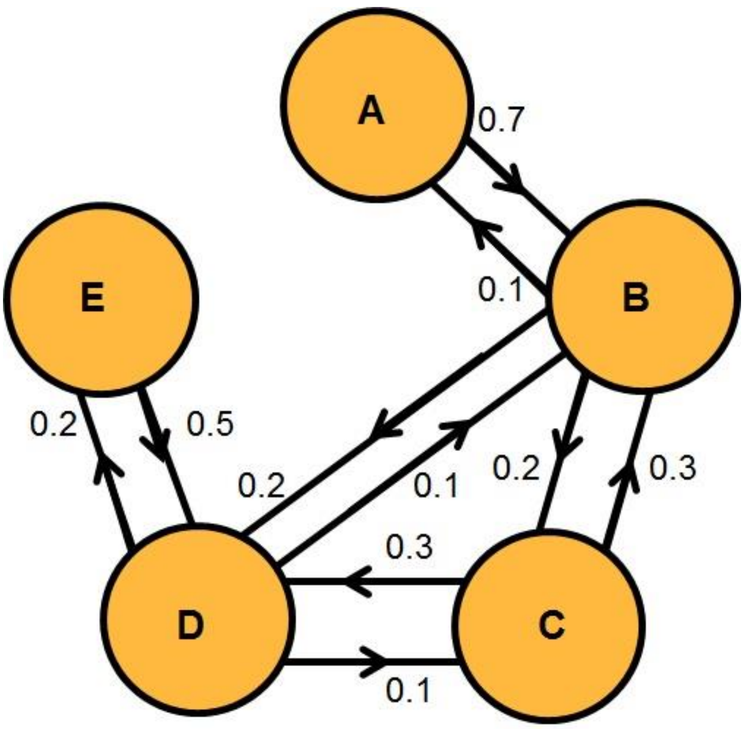
Please hand in your answers, showing all relevant working in the spaces provided. You may use additional sheets for rough working, but only these worksheets should be submitted for assessment.

1.

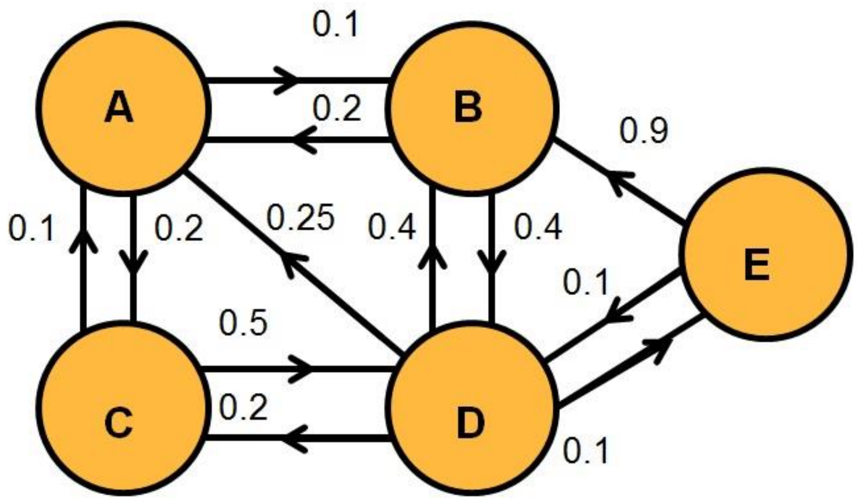
a) Draw the state diagram corresponding to the transition matrix

$$P = \begin{pmatrix} 0.3 & 0.7 & 0 & 0 & 0 \\ 0.1 & 0.5 & 0.2 & 0.2 & 0 \\ 0 & 0.3 & 0.4 & 0.3 & 0 \\ 0 & 0.1 & 0.1 & 0.6 & 0.2 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

clearly labelling all possible transitions with directed arrows weighted with appropriate probabilities.



b) Write down the transition matrix corresponding to the state diagram given below:



$$P = \begin{pmatrix} 0.7 & 0.1 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0.4 & 0 \\ 0.1 & 0 & 0.4 & 0.5 & 0 \\ 0.25 & 0.4 & 0.2 & 0.05 & 0.1 \\ 0 & 0.9 & 0 & 0.1 & 0 \end{pmatrix}$$

(7 Marks)

2. A motorway is assessed each workday and classified as being in one of three states – clear, average or congested.

On 50% of workdays, the state of the road is the same as it was the previous day. On days when the road is clear, it is average the following workday with probability 50%. On days when the road is average, it is either clear or congested on the following workday each with probability 25%. On days when the road is congested, it is either clear or average on the following workday each with probability 25%.

a) Write down the transition matrix for this situation.

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}$$

b) Calculate the 2-step transition matrix (i.e. that which contains the conditional probabilities for the states two days ahead given the present day's state.)

$$P^2 = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}^2 = \begin{pmatrix} 0.375 & 0.5 & 0.125 \\ 0.3125 & 0.4375 & 0.25 \\ 0.3125 & 0.375 & 0.3125 \end{pmatrix}$$

c) i) On a Monday, the road is classified as congested. What is the probability that the road is again congested on Wednesday of the same week.

$$P(\text{Wednesday congested} \mid \text{Monday congested}) =$$

$$\begin{bmatrix} P(\text{Wednesday congested} \mid \text{Tuesday congested}) \\ \times P(\text{Tuesday congested} \mid \text{Monday congested}) \end{bmatrix}$$

$$+\begin{bmatrix} P(\text{Wednesday congested} \mid \text{Tuesday average}) \\ \times P(\text{Tuesday average} \mid \text{Monday congested}) \end{bmatrix}$$

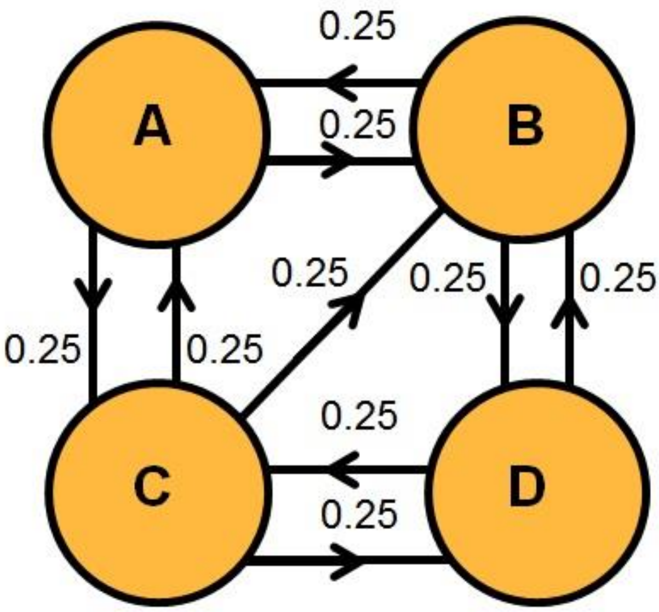
$$+\begin{bmatrix} P(\text{Wednesday congested} \mid \text{Tuesday clear}) \\ \times P(\text{Tuesday clear} \mid \text{Monday congested}) \end{bmatrix}$$

$$= (0.5 \times 0.5) + (0.25 \times 0.25) + (0.25 \times 0) = 0.3125\,.$$

(or could simply read off from the 2-step transition matrix in part b).)

(6 Marks)

3. Consider a Markov Chain with the following state diagram.



a) Find the transition matrix for this Markov Chain.

$$P = \begin{pmatrix} 0.5 & 0.25 & 0.25 & 0 \\ 0.25 & 0.5 & 0 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0.25 & 0.25 & 0.5 \end{pmatrix}$$

b) If the system is observed in state C, what is the probability that it is again in state C three moves later?

$P(X_{n+3} = C | X_n = C)$ can be obtained by working out the probability of all 3-step paths from C to C (e.g. C to C to C to C or C to A to B to C etc.)

Alternatively equating P^3 and picking out the row 3, column 3 value will give the same result.

Here, ${P^3}_{3,3} = 0.171875$.

c) Find the equilibrium distribution for this system.

Solving

$$\begin{pmatrix}\pi_A & \pi_B & \pi_C & \pi_D\end{pmatrix} = \begin{pmatrix}\pi_A & \pi_B & \pi_C & \pi_D\end{pmatrix}\begin{pmatrix}0.5 & 0.25 & 0.25 & 0 \\ 0.25 & 0.5 & 0 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0.25 & 0.25 & 0.5\end{pmatrix}$$

gives

$$\begin{aligned}\pi_A &= 0.5\pi_A + 0.25\pi_B + 0.25\pi_C \\ \pi_B &= 0.25\pi_A + 0.5\pi_B + 0.25\pi_C + 0.25\pi_D \\ \pi_C &= 0.25\pi_A + 0.25\pi_C + 0.25\pi_D \\ \pi_D &= 0.25\pi_B + 0.25\pi_C + 0.5\pi_D\end{aligned}$$

or

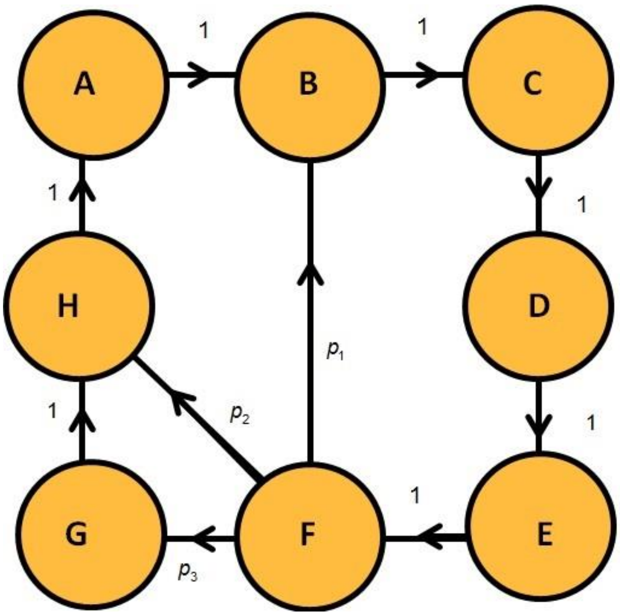
$$\begin{aligned}2\pi_A &= \pi_B + \pi_C \\ 2\pi_B &= \pi_A + \pi_C + \pi_D \\ 3\pi_C &= \pi_A + \pi_D \\ 2\pi_D &= \pi_B + \pi_C\end{aligned}.$$

These can be solved by any scalar multiple of $\begin{pmatrix}3 & 4 & 2 & 3\end{pmatrix}$.

Ensuring $\pi_A + \pi_B + \pi_C + \pi_D = 1$ gives $\begin{pmatrix}\frac{1}{4} & \frac{1}{3} & \frac{1}{6} & \frac{1}{4}\end{pmatrix}$.

(9 Marks)

4. Consider a Markov Chain with the following state diagram.



What conditions must be satisfied by p_1, p_2 and p_3 if:

a) There is an absorbing state?;

Only F can be absorbing and this happens if there are no (non-zero weighted) arrows out of it, so we need $p_1 = p_2 = p_3 = 0$ for this to happen.

b) All states have period 8?;

All states have period 8 only if $p_1 = p_2 = 0$ and $p_3 = 1$. This gives all states certainly returning to themselves after exactly 8 moves.

c) All states with period >1 have period 7?

All states with period >1 have period 7 only if $p_1 = p_3 = 0$ and $p_2 = 1$. This gives all states except G certainly returning to themselves after exactly 7 moves. G is then aperiodic.

d) All states are aperiodic?;

All states are aperiodic if $0 < p_1 < 1$ and/or $0 < p_2 < 1$ and/or $0 < p_3 < 1$.

e) All states communicate with all others?;

All states communicate with all others unless there is at least one state from which at least one other state cannot be reached. As long as $p_3 > 0$, all states communicate with all others.

(10 Marks)

5. Written task. Please see the marking rubric on Canvas before attempting this problem.

In your own words, clearly define a Markov Chain.

Give an example of two sequences of random variables, $X_1, X_2, X_3, ...$ and $Y_1, Y_2, Y_3, ...$ both of which take values from the set of states $\{1, 2, 3, 4, 5, 6\}$ such that $X_1, X_2, X_3, ...$ is not a Markov Chain but $Y_1, Y_2, Y_3, ...$ is a Markov Chain.

Define your variables in everyday language, not mathematical notation

A Markov Chain is a sequence of random variables such that the conditional probability of a future observation, given the present observation is exactly the same as the conditional probability of that observation given the present and past values.

Let $Y_1, Y_2, Y_3, ...$ be the outcomes of a sequence of independent rolls of a regular fair six-sided die. This clearly forms a Markov Chain as no additional information is gained by knowing past outcomes.

Let $X_1, X_2, X_3, ...$ be the value of the largest value obtained on the previous two rolls of a sequence of independent rolls of a regular fair six-sided die. This clearly does not form a Markov Chain as our observations also depend on past values.

(3 marks, including assessment for English Language)