University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) – Tutorial/Laboratory 3
TO BE HANDED IN FOR ASSESSMENT

Please hand in your answers, showing all relevant working in the spaces provided. You may use additional sheets for rough working, but only these worksheets should be submitted for assessment.

1. Let *A, B* and *C* be events such that:

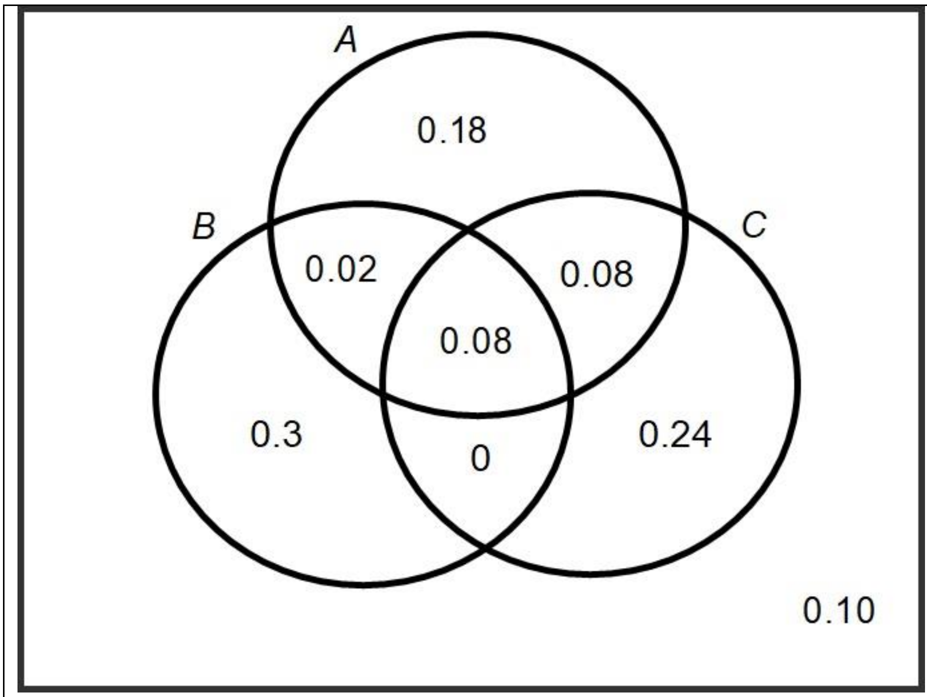
$P(A) = 0.36$, $P(B) = 0.4$ and $P(C) = 0.4$

$P(A \cap B) = 0.1$, $P(A \cap C) = 0.16$ and $P(B \cap C) = 0.08$

$P(A \cap B \cap C) = 0.08$.

a) Calculate the probability that none of the events *A, B* and *C* occur.

Hint: It may help to draw a Venn diagram.



Hence $P(A^c \cap B^c \cap C^c) = 0.10$

(5 Marks)

2. Consider two events A and B such that $P(A \cap B^c) = 0.6$ and $P(A^c \cap B) = 0.3$.

i) Calculate $P(A)$ if $P(A \cup B) = 0.9$;

$P(A \cup B) = P(A \cap B^c) + P(A^c \cap B) + P(A \cap B)$
$0.9 = 0.6 + P(A^c \cap B) + P(A \cap B)$

ii) Explain why A and B cannot partition the sample space;

To form a partition, we need them to be mutually exclusive (i.e. $P(A \cap B) = 0$. If this is the case, $P(A) = 0.6$ and $P(B) = 0.3$, in which case there is a non-zero probability that neither A nor B occurs, so they do not form a partition. To form a partition, we would need $P(A \cup B) = 1$ whereas here is it at most 0.9.

iii) Explain why A and B cannot be independent;

Let $P(A \cap B) = x$ then $P(A) = 0.6 + x$ and $P(B) = 0.3 + x$.
If independent, $P(A \cap B) = P(A)P(B)$ hence we need
$x = (0.6 + x)(0.3 + x)$ or $x^2 - 0.1x + 0.18$.
This has no real solutions.

(4 Marks)

3. Let X be a random variable with probability mass function

$$P(X = k) = \begin{cases} 0.1 & k = -2 \\ 0.4 & k = -1 \\ 0.3 & k = 1 \\ 0.2 & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

i) Calculate $E(X)$;

$E(X) = 0.1(-2) + 0.4(-1) + 0.3(1) + 0.2(2) = 0.1$

ii) Calculate $Var(X)$;

$Var(X) = E(X^2) - E(X)^2$
$= 0.1(-2)^2 + 0.4(-1)^2 + 0.3(1)^2 + 0.2(2)^2 - 0.1^2 = 0.14$

iii) Write down the probability mass function of $X^2 + \sqrt{2}$

$$P((X^2 + \sqrt{2}) = k) = \begin{cases} 0.3 & k = 4 + \sqrt{2} \\ 0.7 & k = 1 + \sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

(5 Marks)

4. For each of the following random experiments, write down the sample space:
Note: Be precise about the use of brackets - (, { or [- and where sets contain infinitely many elements.

i) Rolling a fair die repeatedly and counting how many rolls until it first lands 6;

$\{1,2,3,...\}$ (or $\mathbb{Z}^+$ or $\mathbb{N}$)

ii) Rolling a fair die repeatedly and counting how many rolls until it the total of all rolls first reaches 100 or more;

$\{17,18,19,...,100\} = [17,100] \cap \mathbb{Z}$

iii) Rolling a fair die repeatedly and counting how many rolls until it the total of all rolls is first an even number;

$\{1,2,3,...\}$ (or $\mathbb{Z}^+$ or $\mathbb{N}$)

iv) Rolling a fair die and measuring how far (in mm) from the thrower's hand the die comes to rest, assuming it remains on a table, such that all points on the table are no more than 2m from the thrower's hand, and that this measurement is <u>rounded to the nearest mm</u>;

$\{0,1,2,...,1999,2000\} = \mathbb{Z} \cap [0,2000]$

v) Rolling a fair die and measuring how far (in mm) from the thrower's hand the die comes to rest, assuming it remains on a table, such that all points on the table are no more than 2m from the thrower's hand and that this measurement is <u>not rounded to the nearest mm</u>;

$[0,2000] \cap \mathbb{R}$.

(6 Marks)

5. A box contains four coins. Two are known to be fair and two are known to have Tails on both sides.

a) One coin is selected at random, with all equally likely to be chosen and flipped repeatedly.

i) Calculate the probability that the selected coin lands Tails on the first two flips;

Let F be the event that a fair coin was chosen.

$$P(TT) = P(TT \mid F)P(F) + P(TT \mid F^c)P(F^c)$$
$$= 1^2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{5}{8}.$$

ii) Calculate the probability that exactly one Tails is obtained on the first two flips;

$$P(TH \cap HT) = P(TH \cap HT \mid F)P(F) + P(TH \cap HT \mid F^c)P(F^c)$$
$$= 2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) + (0)^2\left(\frac{1}{2}\right) = \frac{1}{4}.$$

iii) Show that the conditional probability that a fair coin was used if the selected coin lands Tails on the first two flips is 0.8;

$$P(F \mid TT) = \frac{P(F \cap TT)}{P(TT)} = \frac{1^2\left(\frac{1}{2}\right)}{1^2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)} = \frac{\left(\frac{1}{2}\right)}{\left(\frac{5}{8}\right)} = \frac{4}{5}.$$

b) Now, after each flip, the coin is returned to the box and one of the four coins is selected at random as before for the subsequent flip.

i) Calculate the probability that the selected coin lands Tails on the first two flips;

Each flip is now independent of previous flips.

$$P(T) = P(T \mid F)P(F) + P(T \mid F^c)P(F^c) = 1\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{3}{4}.$$

As each flip lands Tails with probability $\frac{3}{4}$, the probability of Tails twice in a row is $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$.

ii) Calculate the probability that exactly one Tails is obtained on the first two flips;

By the same argument as above $P(T) = \frac{3}{4}$ hence $P(TH) = P(HT) = \left(\frac{3}{4}\right)\left(\frac{1}{4}\right)$, so $P(TH \cap HT) = 2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right) = \frac{3}{8}$.

(10 Marks)

6. Written task. Please see the marking rubric on UTSOnline before attempting this problem.

In your own words, clearly explain what it means for a set of events to be independent. Give an example of <u>one</u> random experiment with three different events *X*, *Y* and *Z* such that *X* and *Y* are independent but *Y* and *Z* are not independent.

Clearly state your choice of experiment and state any assumptions you have made.

Note: Ensure that your *X*, *Y* and *Z* are events and not random variables.

Consider rolling one fair six-sided die once.

Let X be the event that the number shown is a 1 or a 2.
Let Y be the event that the number shown is an odd number.
Let Z be the event that the number shown is 1, 2 or 3.

X and Y are independent, since Y occurs 50% of the time and Y also occurs 50% of the time that X occurs. Y and Z are not independent, since if we know we have an odd number, this would increase our belief that the number shown is a 1, 2 or a 3 (since 2/3 of these values are odd.)

(5 marks, including assessment for English Language)