University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) – Tutorial/Laboratory 5
SOLUTIONS

1. A regular fair six-sided die is rolled repeatedly. Using the outcomes of these die rolls, several random variables are defined. Write down the distribution of these. For example, your answer might be of the form "$Z \sim Bin(5,0.5)$" or "Z is a binomial variable with parameters 5 and 0.5"

i) Q is the number of 6s observed on the first roll;

$$Q \sim Bern\left(\frac{1}{6}\right)$$

ii) R is the number of rolls until the first even number is seen;

$$R \sim Geo\left(\frac{1}{2}\right)$$

iii) S is an indicator variable which is 1 if the first four rolls all show the same number and 0 otherwise;

$$S \sim Bern\left(\frac{1}{216}\right)$$

iv) T is the number of 6s shown on the first six rolls;

$$T \sim Bin\left(6,\frac{1}{6}\right)$$

v) U is the number of rolls until the first number greater than 1 is seen, if all rolls landing 6 do not count.

$$U \sim Geo\left(\frac{4}{5}\right)$$

(5 Marks)

2. Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} k & x \in [-7,-2] \cup [2,7] \\ 0 & \text{otherwise} \end{cases}.$$

i) Show that $k = 0.1$.

$$\int_{-\infty}^{\infty} f(x)dx = 1 \text{ hence } \int_{-7}^{-2} kdx + \int_{2}^{7} kdx = 1.$$

$$[kx]_{-7}^{-2} + [kx]_{2}^{7} = 10k = 1$$

This gives $k = 0.1$.

ii) Calculate the expectation of X. Justify your answer.

$E(X) = 0$ since the density function is symmetric about 0.

This can also be established by showing that

$$\int_{-\infty}^{\infty} f(x)dx = 1 \text{ hence } \int_{-7}^{-2} kxdx + \int_{2}^{7} kxdx = \left[\frac{kx^2}{2}\right]_{-7}^{-2} + \left[\frac{kx^2}{2}\right]_{2}^{7} = 0$$

iii) Calculate the variance of X. Justify your answer.

$Var(X) = E(X^2) - E(X)^2 = E(X^2)$ since $E(X) = 0$.

$$Var(X) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{-7}^{-2} kx^2dx + \int_{2}^{7} kx^2dx$$

$$= 1 = \left[\frac{kx^3}{3}\right]_{-7}^{-2} + \left[\frac{kx^3}{3}\right]_{2}^{7} = \left[\frac{7^3 - 2^3}{3}\right] = \frac{335}{3}$$

iv) Calculate $P(X > -3 \mid X > 0)$

$P(X > -3 \mid X > 0) = 1$ since all values greater than 0 are also greater than -3. (This would be true for all possible variables, not just X as defined here.)

v) Calculate $P(X > 4 \mid X > 1)$

$$P(X > 4 \mid X > 1) = \frac{P(X > 4 \cap X > 1)}{P(X > 1)} = \frac{0.3}{0.5} = 0.6.$$

vi) Calculate the cumulative density function of X.

$$F(X) = P(X \le x) = \begin{cases} 0 & x < -7 \\ 0.1(x+7) & x \in [-7,-2] \\ \frac{1}{2} & x \in [-1,1] \\ \frac{1}{2} + 0.1(x-2) & x \in [2,7] \\ 1 & x > 7 \end{cases}$$

(10 Marks)

3. Let $X \sim Bern(0.25)$, $Y \sim Bin(5,0.25)$ and $Z \sim Geo(0.25)$ be independent discrete random variables.

Giving all of your answers in <u>exact form</u> (not a decimal approximation), calculate:

i) $P(Z = 2)$

$P(Z = 2) = (0.25)(0.75)$

ii) $P(Z = 2 \,|\, Y = 3)$

$P(Z = 2) = (0.25)(0.75)$.
Since Z and Y are independent.

iii) $E(X + Z)$

$$E(X + Z) = E(X) + E(Z) = 0.25 + \frac{1}{0.25} = 4.25 .$$

iv) $P(X = 2)$

$P(X = 2) = 0$

v) $P(X - Y = 0)$

$P(X - Y = 0) = P(X = 0)P(Y = 0) + P(X = 1)P(Y = 1)$
$= (0.75)(0.75)^5 + 5(0.25)(0.25)(0.75)^4$

vi) The distribution of X^2

$P(X^2 = 1) = P(X = 1) = 0.25$ and
$P(X^2 = 0) = P(X = 0) = 0.75$ hence $X \sim Bern(0.25)$

vii) The distribution of $X + Y$

Y can be considered as the sum of 5 independent $Bern(0.1)$ variables hence $X + Y$ can be considered as the sum of 11 independent $Bern(0.25)$ variables.

$X + Y \sim Bin(6,0.25)$

(7 Marks)

4. A box contains two six-sided dice. Both are fair (in the sense that all six faces will be shown with equal probability.) One die is numbered as usual i.e. each of the integers 1 to 6. The other die has no 6, but has the number 2 twice instead.
A player selects one die at random, with both equally likely to be selected, and rolls that die repeatedly.

i) Calculate the probability that the chosen die lands 2 on both of the first two rolls.

Let R be the event that the regular die is selected.

$$P(22) = P(22 \mid R)P(R) + P(22 \mid R^c)P(R^c)$$

$$P(22) = \left(\frac{1}{6}\right)^2 \left(\frac{1}{2}\right) + \left(\frac{2}{6}\right)^2 \left(\frac{1}{2}\right) = \frac{5}{72}.$$

ii) Calculate the probability that the chosen die lands 2 on at least one of the first three rolls.

The die lands 2 on at least one of the first three rolls <u>unless</u> all three rolls are not 2s. Let *None* be the event that none of the first three rolls land on a 2.

$$P(None) = P(None \mid R)P(R) + P(None \mid R^c)P(R^c)$$

$$P(None) = \left(\frac{5}{6}\right)^3 \left(\frac{1}{2}\right) + \left(\frac{4}{6}\right)^3 \left(\frac{1}{2}\right) = \frac{189}{2(216)} = \frac{7}{16}$$

The probability of 2 showing at least once is therefore $1 - \frac{7}{16} = \frac{9}{16}$

iii) If the die is rolled three times and 2 is never seen, show that the probability that the regular (numbered 1-6) die is used is $\approx 66.1\%$

$$P(R \mid None) = \frac{P(R \cap None)}{P(None)} = \frac{\left(\frac{5}{6}\right)^3 \left(\frac{1}{2}\right)}{\left(\frac{5}{6}\right)^3 \left(\frac{1}{2}\right) + \left(\frac{4}{6}\right)^3 \left(\frac{1}{2}\right)} = \frac{125}{189} \approx 66.1\%$$

iv) Calculate the fewest times the selected die can be rolled such that, given the observed sequence of outcomes, the probability of having selected the regular die is below 1%.

Hint: Think which outcomes give the strongest evidence against having the regular die

A longer sequence of 2s gives the strongest evidence of having the irregular die.

The probability of using the regular die given $n \in \mathbb{Z}^{+}$ 2s in a row is

$$P(R \mid n\ 2s) = \frac{\left(\frac{1}{6}\right)^{n}\left(\frac{1}{2}\right)}{\left(\frac{1}{6}\right)^{n}\left(\frac{1}{2}\right)+\left(\frac{2}{6}\right)^{n}\left(\frac{1}{2}\right)} = \frac{6^{-n}}{6^{-n}+(6^{-n})2^{n}} = \frac{1}{1+2^{n}}.$$

Solving $\frac{1}{1+2^{n}} < 0.01$ gives $1 < 0.01 + 0.01(2^{n})$ hence $\frac{1-0.01}{0.01} < 2^{n}$. This simplifies to $99 < 2^{n}$, implying that we need at least 7 rolls, since this is the smallest integer n such that $99 < 2^{n}$.

v) What the fewest times the selected die can be flipped such that, given the observed sequence of outcomes, the probability of having <u>not</u> selected the regular is below 1%. Justify your answer.

We can be certain that we have selected the regular after only one roll, if that roll shows a 6.

(8 Marks)

5. Written task. Please see the marking rubric on Canvas before attempting this problem.

In your own words, clearly explain why, for two independent discrete random variables X_1 and X_2 which have the same distribution, there is a non-zero probability that X_1 and X_2 are equal.

Explain why, for two independent continuous random variables Y_1 and Y_2 which have the same distribution, there is effectively a zero probability that Y_1 and Y_2 are equal.

continuous random

Explain your answer in everyday language, not mathematical notation.

X_1 and X_2 take values from a sample space which contains countably many points, each with non-zero probability. Once X_1 has been observed to take a given value, there is a non-zero probability that X_2 takes the same value, hence that both are equal.

As Y_1 and Y_2 are continuous, there is an uncountable number of points that each can take. Once Y_1 has been observed to take a given value, the probability that Y_2 takes the exact same value (with infinitesimal precision) is effectively zero, since the probability of a continuous random variable hitting a given value exactly is infinitesimal, effectively zero.

(5 marks, including assessment for English Language)