University of Technology Sydney
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Probability and Random Variables (37161) – Tutorial/Laboratory 6
SOLUTIONS

1.

a) $$E(X)=\int_{-\infty}^{\infty} xf(x)dx=\int_{-4}^{0} -\frac{x^2}{10}dx+\int_{0}^{2}\frac{x^2}{10}dx$$

$$=\left[-\frac{x^3}{30}\right]_{-4}^{0}+\left[\frac{x^3}{30}\right]_{0}^{2}=-\frac{64}{30}+\frac{8}{30}=-\frac{28}{15}$$

i) $$Var(X)=E(X^2)-\left(\frac{-28}{15}\right)^2.$$

$$E(X^2)=\int_{-\infty}^{\infty} x^2f(x)dx=\int_{-4}^{0} -\frac{x^3}{10}dx+\int_{0}^{2}\frac{x^3}{10}dx=\left[\frac{-x^4}{40}\right]_{-4}^{0}+\left[\frac{x^4}{40}\right]_{0}^{2}=\frac{68}{10}$$

hence $Var(X)=\frac{68}{10}-\left(\frac{-28}{15}\right)^2=\frac{746}{225}$.

ii) $P(X<0\,|\,X^2>4)=1$ since $X^2>4$ implies that either $X>2$ or $X<-2$. The first of these options is impossible (outside the range of *X*), so $X<-2$ hence $X<0$;

iii) $$P(X<-2\,|\,X<0)=\frac{\int_{-4}^{-2} f(x)dx}{\int_{-4}^{0} f(x)dx}=\frac{\left[\frac{-x^2}{20}\right]_{-4}^{-2}}{\left[\frac{-x^2}{20}\right]_{-4}^{0}}=\frac{12}{16}=\frac{3}{4}.$$

2.

i) The range of *Y* is $[0,\infty)$ hence taking the square root of values on this range gives that *Z* also has range $[0,\infty)$.

ii) $Z=\sqrt{Y}$ hence $Y=Z^2$ and $\frac{dY}{dZ}=2Z$.

$$g(y)=\begin{cases}\lambda e^{-\lambda y} & y\in[0,\infty)\\ 0 & \text{otherwise}\end{cases}\quad \text{hence}\quad g(y)=\begin{cases}2\lambda z e^{-\lambda z^2} & z\in[0,\infty)\\ 0 & \text{otherwise}\end{cases}$$

iii) $$\int_{-\infty}^{\infty} g(z)dz=\int_{0}^{\infty}2\lambda z e^{-\lambda z^2}dz=\left[-e^{-\lambda z^2}\right]_{0}^{\infty}=1$$

3. i) $E(T)=6$ and $E(S)=6$.

ii) $P(S=T)=0$ since is it impossible (probability zero) both the first 2 and the first 6 are both obtained on the same roll.

iii) If we are told that the first roll shows a 6, then $S=1$. Knowing that we have already taken one roll and not obtained a 2, the expected number of rolls to obtain the first two is 1 (acknowledging the first roll which landed 6) plus the number of rolls we expect to make until we first see a 2.

4.

i) The range of $W \sim \exp(\lambda)$ is $[0,\infty)$. The transformation $Z=W^{\frac{1}{\gamma}}$ is monotonic with $0^{\frac{1}{\gamma}}=0$ and $W^{\frac{1}{\gamma}} \to \infty$ as $W \to \infty$ so the range of Z is also $[0,\infty)$.

ii) W has density function $f(w)=\begin{cases} \lambda e^{-\lambda w} & w \in [0,\infty) \\ 0 & \text{otherwise} \end{cases}$.

For the transformation $Z=W^{\frac{1}{\gamma}}$, $W=Z^{\gamma}$ and hence $\dfrac{dW}{dZ}=\gamma Z^{\gamma-1}$.

The density function of Z is therefore

$$f(z)=\begin{cases} \lambda\gamma z^{\gamma-1}e^{-\lambda z^{\gamma}} & z \in [0,\infty) \\ 0 & \text{otherwise} \end{cases}.$$

iv) Comparing this to the density function of $T \sim Weibull(\beta,\eta)$ (given), we see that $Z \sim Weibull\left(\gamma,\left(\dfrac{1}{\lambda}\right)^{\frac{1}{\gamma}}\right)$

We see this by substituting $\beta=\gamma$ and $\eta=\left(\dfrac{1}{\lambda}\right)^{\frac{1}{\gamma}}$ into the density function for $T \sim Weibull(\beta,\eta)$.