University of Technology Sydney
School of Mathematical and Physical Sciences

Probability and Random Variables (37161) – Tutorial/Laboratory 7
SOLUTIONS

1. A company sells tickets to a show. All tickets are either full price tickets or concession tickets. Each ticket is valid for either one afternoon show or one evening show.
The sale of each possible ticket type is modelled with a Poisson process, with the sale of each ticket assumed to be independent of all other sales and ticket types.

The expected number of full price tickets sold per day is 60.
The expected number of concession tickets sold per day is 30.
It is expected that 25% of tickets sold will be for afternoon shows and 75% of tickets sold will be for evening shows.
Assume sales are processed 24 hours a day at the same expected rates.

Calculate the probabilities of the following events, leaving your answer in exact form not a decimal approximation. For example, if your answer is $\frac{e^{-2}2^3}{3!}$, please leave it in this form and do not approximate it to a decimal.

i) In one hour, no concession tickets are sold;

> Concession sales per hour $\sim Poi\left(\frac{30}{24}\right)$ hence the probability of zero of these sales is $e^{-\frac{5}{4}}$.

ii) In one hour, no concession tickets are sold to evening shows;

> Evening concession sales per hour $\sim Poi\left(\frac{30}{24}\times 0.75\right)$ hence the probability of zero of these sales is $e^{-\frac{15}{16}}$.

iii) In ten minutes, exactly two tickets are sold;

> Total sales per 10 mins $\sim Poi\left(\frac{60+30}{24}\frac{1}{6}\right)$ hence the probability of two of these sales is $\frac{e^{-\frac{5}{8}}\left(\frac{5}{8}\right)^2}{2!}$.

iv) The first ticket sale recorded is a concession ticket to an afternoon show;

> Expected sales per day is 90. Of these, the expected number which are concession tickets for afternoon shows is 30×0.25 hence the probability that the next ticket is a concession ticket to an afternoon show is $\frac{30 \times 0.25}{90} = \frac{1}{12}$.

v) More than one hour passes between successive sales of afternoon tickets;

> Afternoon sales per hour $\sim Poi\left(\frac{60+30}{24} \times 0.25\right)$ hence the probability of zero of these sales in an hour is $e^{-\frac{15}{16}}$.

vi) In one hour, no concession tickets are sold, given that 5 full price tickets are sold.

> The sales of concession and full price tickets are independent, hence the conditional information adds nothing. As with part i) we have a probability of $e^{-\frac{5}{4}}$.

(6 Marks)

2. Consider drawing cards repeatedly at random from a standard deck of 52 cards, with all cards equally likely to be picked each time and with cards returned to the deck and reshuffled after each selection.
A player begins with 0 points.
If he/she selects a Hearts card, he/she adds 1 point to the score and the game ends.
If he/she selects a Diamonds card, he/she adds 2 points to the score and the game ends.
If he/she selected a Clubs or Spades card, he/she adds 3 points to the score and the game continues.

Let X_i be the points added on the i^{th} card selection where $i \in \{1,2,3,...\}$.
Let Y be the number of additional points scored until the game ends.

i) Write down the probability mass function of X_1.

$$P(X_1 = k) = \begin{cases} 0.25 & k=1 \\ 0.25 & k=2 \\ 0.5 & k=3 \\ 0 & \text{otherwise} \end{cases}$$

ii) Hence show that $P(X_1 < E(X_1)) = 0.5$.

$E(X_1) = 1(0.25) + 2(0.25) + 3(0.5) = 2.25$.
$P(X_1 < 2.25) = P(X_1 = 1) + P(X_1 = 2) = 0.5$

iii) Here, we have that
$E(Y) = 0.25E(Y \mid X_i = 1) + 0.25E(Y \mid X_i = 2) + 0.5E(Y \mid X_i = 3)$.
Apply the law of total expectation to show that
$E(Y) = 2.25 + 0.5E(Y)$

$E(Y) = 0.25E(Y \mid X_i = 1) + 0.25E(Y \mid X_i = 2) + 0.5E(Y \mid X_i = 3)$
Taking the expectation of both sides gives
$E(Y) = 0.25(1) + 0.25E(2) + 0.5\left(3 + E(Y)\right)$. This simplifies to
$E(Y) = 2.25 + 0.5E(Y)$.

iv) Hence calculate $E(Y)$

$E(Y) = 2.25 + 0.5E(Y)$ so $E(Y) = \dfrac{2.25}{0.5} = 4.5$.

v) In your own words, clearly explain why $Var(Y \mid X_i = 1) = 0$

If a Hearts card is selected, then the game ends with 1 additional point. There is no uncertainty (i.e. variance 0) about the number of additional points scored.

vi) Write down $Var(Y \mid X_i = 2)$. Justify your answer.

For the same reasons as above, selecting a Diamonds card ends the game with 2 additional points. There is no uncertainty about this.

vii) Calculate $Var(Y \mid X_i = 3)$ in terms of $Var(Y)$. Justify your answer.

Selecting Clubs or Spades adds 3 points, but does not bring the game any closer to conclusion, hence we still have the same uncertainty as before. $Var(Y \mid X_i = 3) = Var(Y+3) = Var(Y)$.

viii) Using your answers to parts v), vi) and vii), show that $E(Var(Y \mid X_i)) = 0.5Var(Y)$

$$\begin{aligned} E(Var(Y \mid X_i)) &= Var(Y \mid X_i = 1)P(X_i = 1) \\ &\quad + Var(Y \mid X_i = 2)P(X_i = 2) + Var(Y \mid X_i = 3)P(X_i = 3) \\ &= 0.25(0) + 0.25(0) + 0.5Var(Y) \\ &= 0.5Var(Y) \end{aligned}$$

ix) Show that $E(E(Y^2 \mid X_i)) = \frac{235}{8}$

$E(Y^2 \mid X_i = 1) = 1^2$, $E(Y^2 \mid X_i = 2) = 2^2$ and

$E(Y^2 \mid X_i = 3) = (3 + E(Y))^2 = 7.5^2$.

$$E(E(Y^2 \mid X_i)) = 1(0.25) + 4(0.25) + 7.5^2(0.5) = \frac{235}{4}.$$

x) Using parts iv) and ix) Show that $Var(E(Y \mid X_i)) = \frac{73}{8}$.

$$Var(E(Y \mid X_i)) = E(E(Y^2 \mid X_i)) - E(E(Y \mid X_i))^2 = \frac{235}{8} - 4.5^2 = \frac{73}{8}.$$

xi) Hence apply the law of total variance $Var(Y) = E(Var(Y \mid X_i)) + Var(E(Y \mid X_i))$ to find $Var(Y)$.

Combining parts viii) and x) gives

$Var(Y) = 0.5Var(Y) + \frac{73}{8}$ hence $Var(Y) = \frac{73}{4}$.

(18 Marks)

3. Let $X \sim U[0,1]$ be a continuous uniform random variable.

That is, X has density function $f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$.

Consider the change of variable $Y = 4X + 7$.

i) Calculate X in terms of Y.

> $Y = 4X + 7$ hence $X = \dfrac{Y-7}{4}$.

ii) Calculate $\dfrac{dY}{dX}$.

> $\dfrac{dY}{dX} = 4$.

iii) What is the range of Y? Justify your answer.

> $Y \in [7,11]$ since the smallest Y can be is $4(0)+7=7$ and the largest it can be is $4(1)+7=11$

iv) Hence show that Y is also a continuous uniform random variable, $Y \sim U[a,b]$ for some $a < b$. Clearly state the values of a and b.

> The change of variable gives that the density function of Y is $f(x(y)) \times \dfrac{dx}{dy} = 1 \times \dfrac{1}{4}$. This is valid between 7 and 11 and zero elsewhere. This gives $f(y) = \begin{cases} \dfrac{1}{4} & y \in [7,11] \\ 0 & \text{otherwise} \end{cases}$ hence $Y \sim U[7,11]$

(6 Marks)

4. Written task. Please see the marking rubric on Canvas before attempting this problem.

In your own words, clearly explain why, for two independent variables X and Y, the conditional variance of $X \mid Y$ is the same as the variance of X.

Explain further why, if X and Z, are not independent, the conditional variance of $X \mid Z$ must be less than the variance of X.

If you wish to give an example to explain your answer, please do so in everyday language, not mathematical notation.

If X and Y are independent, then observing the value of Y tells us nothing about the possible value of X, hence does not alter our uncertainty about its value. This means the conditional variance is the same as the unconditional variance.

If X and Z are not independent, then observing the value of Z tells us something about the possible value of X. This additional information can only "narrow down" our uncertainty about X, not increase it, hence the conditional variance must be smaller than the unconditional variance.

(5 marks, including assessment for English Language)