University of Technology Sydney
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Probability and Random Variables (37161) – Tutorial/Laboratory 8

1.

a) Let X be the number shown when rolling one regular fair six-sided die.

i) Calculate the generating function of X, $g_X(z) = E(z^X)$.

ii) By differentiating $g_X(z)$, show that $E(X) = 3.5$.

b) A discrete random variable Y has generating function

$$g_Y(z) = \frac{1}{2} + \frac{1}{3}z^{10} + \frac{1}{6z^3}.$$

i) By differentiating $g_Y(z)$, find $E(Y)$.

ii) Write down the probability mass function of Y, $P(Y = k)$ for all possible k.

c) A discrete random variable Q has generating function

$$g_Q(z) = e^{10(z-1)}.$$

i) By differentiating $g_Q(z)$, find $E(Q)$.

ii) Write down the probability mass function of Q, $P(Q = k)$ for all possible k.

d) The continuous uniform random variable $V \sim U[1,11]$ has probability density function

$$f(v) = \begin{cases} 0.1 & v \in [1,11] \\ 0 & \text{otherwise} \end{cases}.$$

Show that the generating function of V, is

$$g_V(z) = E(z^V) = \int_1^{11} z^v f(v)\,dv = \frac{z^{11} - z}{10\ln(z)}.$$

Hint: Consider writing z^v as an exponential with base e.

2.

a) An exponential random variable $W \sim \exp(\lambda)$ has probability density function

$$f(w) = \begin{cases} \lambda e^{-\lambda w} & w \in [0,\infty) \\ 0 & \text{otherwise} \end{cases}.$$

i) Showing all of your working, calculate the generating function of W, $g_W(z) = E(z^W)$.

ii) By differentiating $g_W(z)$, find $E(W)$.

b) A geometric random variable $R \sim Geo(p)$ has probability mass function

$$P(R = r) = \begin{cases} p(1-p)^{r-1} & r \in \{1,2,3,...\} \\ 0 & \text{otherwise} \end{cases}.$$

i) Calculate the generating function of R.

ii) Show that, if $W_1, W_2, W_3, ...$ are independent random variables such that each $W_i \sim \exp(\lambda)$ and if $R \sim Geo(p)$, then $S = \sum_{i=1}^{R} W_i$ then S is exponentially distributed. State its rate parameter.

Hint: You may use without proof the result that $g_S(z) = g_R(g_{W_i}(z))$.

If a fixed number of independent identically-distributed exponential variables are summed, the resulting variable is an Erlang variable.

That is, if $W_1, W_2, W_3, ... W_N$ each $\sim \exp(\lambda)$ then for $\left(W_1 + W_2 + W_3 + ... + W_N\right) = T$, $T \sim Erlang(N, \lambda)$

c) By using your answer to part i) calculate the generating function of $T \sim Erlang(N, \lambda)$

Note: An Erlang variable describes the time between non-successive arrivals in a Poisson process. For example, if the number of calls into a telephone call centre in an hour $\sim Poi(\lambda)$ then the time between successive arrivals (for example between the first and second call recorded) $\sim \exp(\lambda)$.

The time between, say, the first and the tenth call is the sum of the time between the first and the second plus between the second and third, plus between the third and fourth etc. The total of these nine successive interarrival times $\sim Erlang(9, \lambda)$.

3.

If $X \sim Bin(n, p_1)$ and $T \sim Bern(p_2)$ then

$g_X(z) = E(z^X) = \left[(1-p_1) + p_1 z\right]^n$ .and $g_T(z) = E(z^T) = \left[(1-p_2) + p_2 z\right]$

Let $T_1, T_2, ..., T_X$ each be independent variables such

that each $T_i \sim Bern(p_2)$ and where $X \sim Bin(n, p_1)$. Let $S = \sum_{i=0}^{X} T_i$.

Show that S is a binomial random variable and find both its parameters.

Hint: You may use without proof the result that $g_S(z) = g_X(g_{T_i}(z))$.

4.

500 players enter a competition. Players born on a weekend (Saturday or Sunday) are instantly eliminated. All remaining players flip a fair coin once. (Assume each person, independent of all others, is equally likely to be have been born on each day of the week.)
If a player's coin lands Heads, he/she is eliminated from the game. If it lands Tails, he/she then rolls a regular fair six-sided die.

If the die lands on a 5 or a 6, he/she is eliminated from the game. If it lands on a 1, 2, 3 or 4, he/she then selects one card at random from a regular deck of 52 playing cards (each player has his/her own deck.)
If the card selected is an even (2, 4, 6, 8 or 10) numbered card, he/she is eliminated from the game. If it is any other card, he/she is declared a winner.

The number of players winning the game, W_T , is therefore

$W_T = \sum_{i=0}^{D_T} W_i$ where W_i is number of winning cards selected (i.e. 0 or 1) from the ith player's deck and D_T is the total number of remaining players who select a card

and where

$D_T = \sum_{i=0}^{C_T} D_i$ where D_i is number of winning rolls (i.e. 0 or 1) from the ith player's die roll and C_T is the total number of remaining players who get a roll a die

and where

$C_T = \sum_{i=0}^{B_T} C_i$ where C_i is number of coin flips landing Tails (i.e. 0 or 1) from the ith player's coin and B_T is the total number of remaining players who get a roll a die

and where

$B_T = \sum_{i=0}^{500} B_i$ where B_i is number of weekday birthdays (i.e. 0 or 1) for the ith player.

a) Write down the distribution of each of:

i) $B_1, B_2, \ldots$

ii) $C_1, C_2, \ldots$

iii) $D_1, D_2, \ldots$

iv) $W_1, W_2, \ldots$

Hint: Each of these is the same type of variable, albeit with different parameter(s).

b) Write down the distribution of B_T.

c) Find the distribution of W_T.

Hint: Your answer to Question 3 may be helpful with part c)