# University of Technology Sydney
# School of Mathematical and Physical Sciences

Probability and Random Variables (37161) – Tutorial/Laboratory 8
SOLUTIONS

1.

a)

i) $g_X(z) = E(z^X) = \frac{1}{6}z + \frac{1}{6}z^2 + \frac{1}{6}z^3 + \frac{1}{6}z^4 + \frac{1}{6}z^5 + \frac{1}{6}z^6$.

ii) $g'_X(z) = \frac{1}{6} + \frac{2}{6}z + \frac{3}{6}z^2 + \frac{4}{6}z^3 + \frac{5}{6}z^4 + \frac{6}{6}z^5$ hence $E(X) = g'_X(1) = 3.5$

b)

i) $g'_Y(z) = \frac{10}{3}z^9 - \frac{3}{6z^4}$ hence $E(Y) = g'_Y(1) = \frac{10}{3} - \frac{1}{2} = \frac{17}{6}$.

ii)
$$P(Y = k) = \begin{cases} \frac{1}{2} & k = 0 \\ \frac{1}{3} & k = 10 \\ \frac{1}{6} & k = -3 \\ 0 & \text{otherwise} \end{cases}$$

c)

i) $g'_Q(z) = 10e^{10(z-1)}$ hence $E(Q) = g'_Q(1) = 10$.

ii) $Q \sim Poi(10)$ hence
$$P(Q = k) = \begin{cases} \frac{e^{-10}10^k}{k!} & k \in \{0,1,2,3,...\} \\ 0 & \text{otherwise} \end{cases}$$

d)

$$g_V(z) = E(z^V) = \int_1^{11} z^v f(v)dv = \int_1^{11} 0.1e^{v\ln(z)}dv$$

$$= \left[\frac{0.1e^{v\ln z)}}{\ln(z)}\right]_1^{11} = \left[\frac{0.1e^{11\ln z)}}{\ln(z)} - \frac{0.1e^{\ln z)}}{\ln(z)}\right] = \frac{z^{11} - z}{10\ln(z)}$$

2.

a)

i)

$$g_W(z) = E(z^W) = \int_0^\infty z^w \lambda e^{-\lambda w} dw = \int_0^\infty e^{w\ln(z)} \lambda e^{-\lambda w} dw = \int_0^\infty \lambda e^{-w(\lambda - \ln(z))} dw$$

$$= \int_0^\infty \lambda e^{-w(\lambda - \ln(z))} dw = \left[ -\frac{\lambda}{\lambda - \ln(z)} e^{-w(\lambda - \ln(z))} \right]_0^\infty = \frac{\lambda}{\lambda - \ln(z)}.$$

ii) $g_W'(z) = \dfrac{\lambda}{z\left[\lambda - \ln(z)\right]^2}$ hence $E(W) = g_W'(1) = \dfrac{1}{\lambda}$.

b)

i) $g_R(z) = E(z^R) = \sum_{k=0}^{\infty} z^k p(1-p)^{k-1}$

$$= pz + pz(1-p)z + pz(1-p)^2 z^2 + pz(1-p)^3 z^3 + \ldots$$

This is a geometric series, first term $pz$, common ratio $(1-p)z$. Hence (assuming $-1 < (1-p)z < 1$) $g_R(z) = \dfrac{pz}{1-(1-p)z}$.

ii) We know that, for $S = \sum_{i=1}^{R} W_i$, then $g_S(z) = g_R(g_{W_i}(z))$. Here $R \sim Geo(p)$ hence $g_R(z) = \dfrac{pz}{1-(1-p)z}$ and each $W_i \sim \exp(\lambda)$ hence each $g_{W_i}(z) = \dfrac{\lambda}{\lambda - \ln(z)}$.

We therefore have that

$$g_S(z) = g_R(g_{W_i}(z)) = g_R(z) = \frac{p\dfrac{\lambda}{\lambda - \ln(z)}}{1-(1-p)\dfrac{\lambda}{\lambda - \ln(z)}} = \frac{p\lambda}{p\lambda - \ln(z)}$$

so $S \sim \exp(p\lambda)$.

c) If $W_1, W_2, W_3, \ldots$ each $\sim \exp(\lambda)$ then the generating function of the variable obtained by summing $N$ of these is equal to multiplying the generating functions of each of these together. For $\left(W_1 + W_2 + W_3 + \ldots + W_N\right) = T$,

$$g_T(z) = g_{W_1}(z) g_{W_2}(z) \ldots g_{W_N}(z) = \left[g_{W_i}(z)\right]^N = \left(\frac{\lambda}{\lambda - \ln(z)}\right)^N.$$

3.

$$g_S(z) = g_X(g_{T_i}(z)) = \left[(1-p_1) + p_1\left[(1-p_2) + p_2 z\right]\right]^n .$$
$$= \left[(1-p_1 p_2) + p_1 p_2 z\right]^n .$$

hence $S \sim Bin(n, p_1 p_2)$.

4.

a) i) $B_1, B_2, ... \sim Bern\left(\frac{5}{7}\right)$

ii) $C_1, C_2, ... \sim Bern\left(\frac{1}{2}\right)$

iii) $D_1, D_2, ... \sim Bern\left(\frac{2}{3}\right)$

iv) $W_1, W_2, ... \sim Bern\left(\frac{8}{13}\right)$

b) $B_T \sim Bin\left(500, \frac{5}{7}\right)$.

c) $W_T \sim Bin\left(500, \left(\frac{5}{7} \times \frac{1}{2} \times \frac{2}{3} \times \frac{8}{13}\right)\right) \sim Bin\left(500, \frac{40}{273}\right)$.