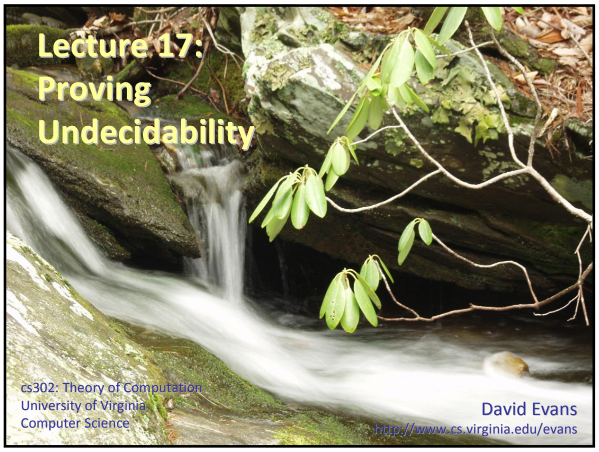


Proofs of *Decidability*

How can you prove a language is *decidable*?

What Decidable Means

A language L is **decidable** if there exists a TM M such that for all strings w:

- If $w \in L$, M enters q_{Accept}.
- If $w \notin L$, M enters q_{Reject}.

To prove a language is decidable, we can show how to construct a TM that decides it.

For a correct proof, need a convincing argument that the TM always eventually accepts or rejects any input.

Proofs of **Un***decidability*

How can you prove a language is ***un****decidable*?

Proofs of Undecidability

To prove a language is *undecidable*, need to show there is **no** Turing Machine that can decide the language.

This is hard: requires reasoning about *all* possible TMs.

Proof by Reduction

X

1. We know X does not exist.
(e.g., X = a TM that can decide A_{TM})

2. Assume Y exists.
(e.g., Y = a TM that can decide B)

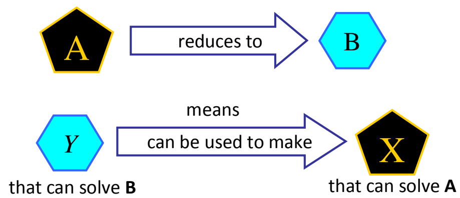
Y

3. Show how to use Y to make X.

Y

4. Since X does not exist, but Y could be used to make X, then Y must not exist.

Reduction Proofs

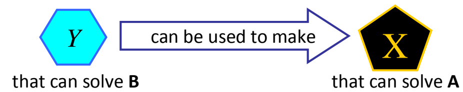


Hence, **A is not a harder problem than B.**

The name "reduces" is confusing: it is in the *opposite* direction of the making.

Converse?

A reduces to B



A is not a harder problem than B.

Does this mean **B** is as hard as **A**?

No! *Y* can be ***any*** solver for **B**. **X** is ***one*** solver for **A**. There might be easier solvers for **A**.

Reduction Pitfalls

- Be careful: the direction matters a great deal
  - Showing a machine that decides B can be used to build a machine that decides A shows that A is not harder than B.
  - To show equivalence, need reductions in both directions.
- The transformation must involve only things you know you ***can do***: otherwise the contradiction might be because something else doesn't exist.

What does can do mean here?

What "Can Do" Means

- The transformations in a reduction proof are limited by what you are proving
- For undecidability proofs, you are proving something about all TMs: the reduction transformations are anything that a TM can do that is guaranteed to terminate
- For complexity proofs (later), you are proving something about how long it takes: the time it takes to do the transformation is limited

The Halting Problem

$HALT_{TM}$ = { <M, w> | M is a TM description and M halts on input w }

Alternate statement as problem:
Input: A TM M and input w
Output: True if M halts on w, otherwise **False**.

Is $HALT_{TM}$ Decidable?

- Possible "Yes" answer: Prove it is decidable

  Design a TM that can decide $HALT_{TM}$

- Possible "No" answer: prove it is undecidable

  Show that **no** TM can decide $HALT_{TM}$

  Show that **a** TM that could decide $HALT_{TM}$ could be used to decide A_{TM} which we already proved is undecidable.

Acceptance Language

A_{TM} = { <M, w> | M is a TM description
and M accepts input w }

We proved A_{TM} is undecidable last class.

Since we know ATM is undecidable, we can show a new language B is undecidable if a machine that can decide B could be used to build a machine that can decide A_{TM}.

Reducing A_{TM} to $HALT_{TM}$

$HALT_{TM}$ = { <M, w> | M is a TM description
and M halts on input w }

A_{TM} = { <M, w> | M is a TM description
and M accepts input w }

<M, w> is in A_{TM} if and only if:
M halts on input w
and when M halts it is in accepting state.

Deciding A_{TM}

- Assume $HALT_{TM}$ is decidable.
- Then some TM R can decide $HALT_{TM}$.
- We can use R to build a machine that decides A_{TM}:
  - Simulate R on <M, w>
  - If R rejects, it means M doesn't halt: **reject**.
  - If R accepts, it means M halts:
    - Simulate M on w, accept/reject based on M's accept/reject.

Since **any** TM that decides $HALT_{TM}$ could be used to build a TM that decides A_{TM} (which we know is impossible) this proves that no TM exists that can decide $HALT_{TM}$.

Equivalence of DFA D and TM M

EQ_{DM} = { <D, T> | D is a DFA description,
T is a TM description
and $L(T) = L(D)$ }

Is EQ_{DM} decidable?

EQ_{DM} Is Undecidable

- Suppose R decides EQ_{DM}.
- Can we use R to decide $HALT_{TM}$?

$HALT_{TM}$ = { <M, w> | M is a TM description
and M halts on input w }

EQ_{DM} = { <D, T> | D is a DFA description,
T is a TM description
and $L(T) = L(D)$ }

Given M and w, how can you construct D and T so R(<D, T>) tells you if M halts on w?

EQ_{DM} Is Undecidable

$HALT_{TM}$ = { <M, w> | M is a TM description
and M halts on input w }

EQ_{DM} = { <D, T> | D is a DFA description,
T is a TM description
and $L(T) = L(D)$ }

D = DFA that accepts all strings.

T = TM that ignores input and simulates M on w, and if simulated M accepts or rejects, accept.

EQ_{DM} Is Undecidable

$HALT_{TM}$ = { <M, w> | M is a TM description and M halts on input w }

EQ_{DM} = { <D, T> | D is a DFA description, T is a TM description and $L(T) = L(D)$ }

~~D = DFA that rejects all strings.~~

~~T = TM that ignores input and simulates M on w, and if simulated M accepts or rejects, reject.~~

Rice's Theorem

Henry Gordon Rice, 1951

Any *nontrivial* property about the language of a Turing machine is undecidable.

Nontrivial means the property is true for *some* TMs, but not for *all* TMs.

Which of these are Undecidable?

- Does TM M accept any strings? Undecidable
- Does TM M accept all strings? Undecidable
- Does TM M accept "Hello"? Undecidable
- Does TM M_1 accept more strings than TM M_2? Undecidable
- Does TM M take more than 1000 steps to process input w? Decidable
- Does TM M_1 take more steps than TM M_2 to process input w? Undecidable

Next Class

- Examples of some problems we actually care about that are undecidable
- Are there any problems that we don't know if they are decidable or undecidable?

- PS5 Due next Tuesday (April 1)
- Exam 2 in two weeks