Week 11 exercise answers

October 17, 2022

For $m \in \mathbb{N}$, $[m] = \{1, 2, \ldots, m\}$.

1. Let $G = (V, E)$ be a graph. Recall that $S \subseteq V$ is a clique, if every pair of vertices in S is connected in G. Let $CLIQUE = \{(G, k) | G \text{ has a size-}k \text{ clique}\}$. Prove that $CLIQUE$ is NP-complete.

   Answer. Hint: reduce IND to $CLIQUE$: take the complement graph.

2. Let U be a finite set. Let $S_1, \ldots, S_k$ be subsets of U. We say that S_i's form a set cover of U, if $\cup_{i=1}^{k} S_i = U$. Let $SETCOVER = \{(U, S_1, \ldots, S_m, k) | \exists T \subseteq [m], |T| \leq k, \cup_{i \in T} S_i = U\}$. Prove that $SETCOVER$ is NP-complete. (Hint: reduce the vertex cover problem to $SETCOVER$.)

   Answer. First show that SETCOVER is in NP.

   Then show that SETCOVER is NP-hard. Let $G = (V, E)$ be a graph. Construct a SETCOVER instance as follows: $U = E$. For $v \in V$, let $S_v = \{e \in E | e \text{ is incident to } v\}$. Then do the following:

   (a) Correctness: G has a vertex cover of size k if and only if there exists a size-k set over in $\{S_v | v \in V\}$ for $U = E$.

   (b) Running time: the construction takes $O(nm)$-time.

3. (a) Recall that a Hamiltonian path in a directed graph G is a path (instead of cycle) that visits each vertex exactly once. And

   $$HAMPATH = \{(G, s, t) | G \text{ is a directed graph with a Hamiltonian path from } s \text{ to } t\}.$$

   Prove that $HAMPATH$ is NP-complete.

   Answer. Use the construction in class, but without an edge from t to s.

   (b) Another problem on directed graphs is as follows. Let

   $$LONGPATH = \{(G, s, t, k) | G \text{ a directed graph with a simple path from } s \text{ to } t \text{ of length } \geq k\}.$$

   Prove that $LONGPATH$ is NP-complete.

   Answer. Hint: $(G, s, t) \in HAMPATH \iff (G, s, t, |V|) \in LONGPATH$.

4. The language Partition consists of all $\{a_1, \ldots, a_n \mid a_i \in \mathbb{Z}$ and there is a subset $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i\}$. Show that Partition is NP-complete.

   Answer. To show that Partition is NP-complete. First one needs to prove Partition is in NP. This can be easily shown because a nondeterministic Turing machine can guess the subset S, and verify that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$ in polynomial time.

Next we show PARTITION by reducing SUBSETSUM to it. Recall that a SUBSETSUM problem instance takes a set of integers $W = \{w_1, \ldots, w_n\}$ and a target integer w as input, and decides whether there exists a subset $S \subseteq W$ such that $\sum_{i \in S} w_i = w$. Given such an input instance, feed $A = W \cup \{s - 2w\}$ into PARTITION, where $s = \sum_i w_i$ is the sum of all integers in W. We will prove that $\langle W, w \rangle \in$ SUBSETSUM $\iff \langle A \rangle \in$ PARTITION.

$\Longrightarrow$: If there exists a set of numbers in W that sum to w, then the remaining numbers in W sum to $s - w$. Therefore, there exists a partition of A into two such that each partition sums to $s - w$.

$\Longleftarrow$: Let's say that there exists a pair of partitions of A such that the sum of in each partition is $s - w$. One of these partitions contains the number $s - 2w$. Removing this number, we get a set of numbers whose sum is w, and all of these numbers are in W.