48510 LEC 10 – PHASOR

Topic 10-1-1: What is phasor?

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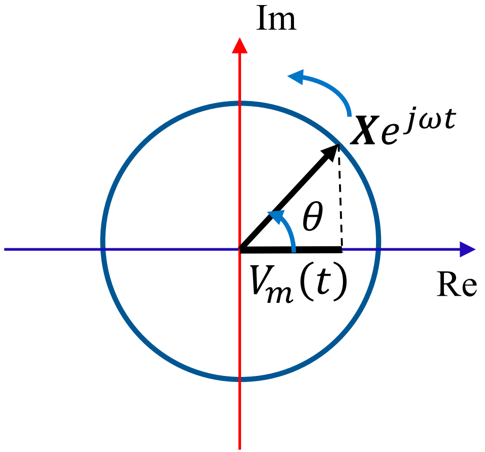
10-1-1 WHAT IS PHASOR

A sinusoidal waveform $V(t) = V_m \cos(\omega t + \theta)$ can be generated by rotating a vector (complex number **X**) in the complex plane and take its projection on the real axis.

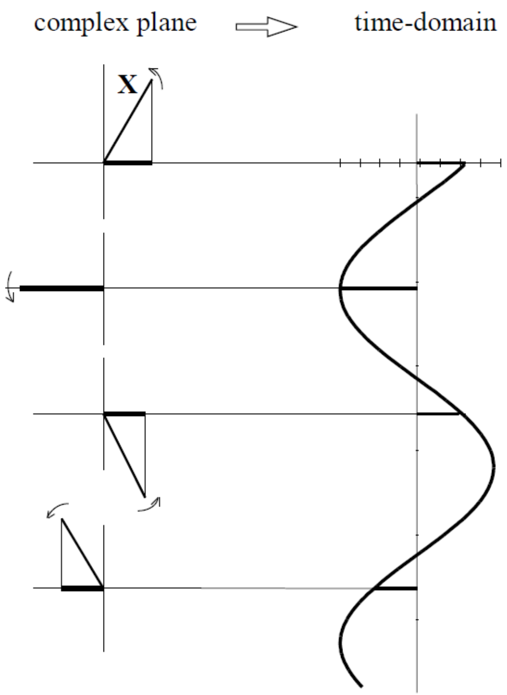


$$V(t) = V_m cos(\omega t + \theta)$$

This is the time domain representation of the waveform.

$$\boldsymbol{X} = V_m e^{j\theta}$$

This is the phasor, the frequency domain representation, of the waveform.



By rotating the vector **X**, we multiply **X** with $e^{j\omega t}$

$$Xe^{j\omega t} = V_m e^{j\theta} e^{j\omega t} = V_m\ e^{j(\omega t+\theta)} = V_m\, cos(\omega t + \theta) + jV_m \sin(\omega t + \theta)$$

$\boldsymbol{e^{j\omega t}}$ *determines how fast we rotate the vector and how fast the wave travels.*

For the phasors, ***the frequency part*** $\boldsymbol{e^{jwt}}$ ***is omitted*** *as the force function (input) and the resultant response (output) will always have the same frequency in a* ***linear*** *circuit.*

10-1-1 WHAT IS PHASOR

Definition of phasors: phasors are complex numbers that represent sinusoidally varying currents or voltages. The magnitude of a phasor equals the peak value of the waveform and the angle equals the phase of the waveform (in cosine convention).

Waveforms

$$V_1(t) = V_1\cos(\omega t + \theta_1)$$

$$V_2(t) = V_2\sin(\omega t + \theta_2)$$
$$= V_2\cos(\omega t + (\theta_2 - 90^0))$$

Phasors

$$V_1\angle\theta_1 = V_1 e^{j\theta_1}$$

$$V_2\angle(\theta_2 - 90^0) = V_2 e^{j\theta_2}$$

Don't forget to always convert sin to cosin!!

$$\underline{\boldsymbol{\sin(\omega t + \theta) = cos(\omega t + \theta - 90^0)}}$$

$\mathbf{X} = X_m\angle\phi$	polar form
$\mathbf{X} = X_m e^{j\phi}$	exponential form
$\mathbf{X} = X_m(\cos\phi + j\sin\phi)$	trigonometric form
$\mathbf{X} = a + jb$	rectangular form

Four forms of phasor

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Topic 10-1-2: Why use phasors?

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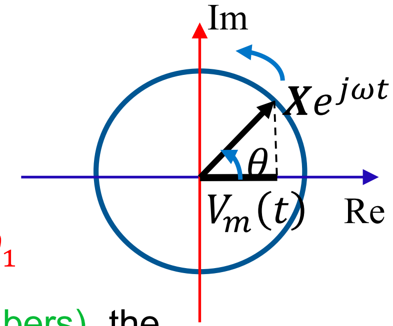
10-1-2 WHY USE PHASORS?



Why use phasors?

For the ease of calculations when analysing AC circuits.

Waveform (time-domain)	*Phasors (frequency-domain)*
$V_1(t) = V_1\cos(\omega t + \theta_1)$	$V_1\angle\theta_1 = V_1 e^{j\theta_1} = V_1 cos\theta_1 + jV_1 sin\theta_1$

By convert the waveforms (time-variant functions) to phasors (fixed complex numbers), the calculation is much easier.

Is it correct to use phasors to replacing waveforms?

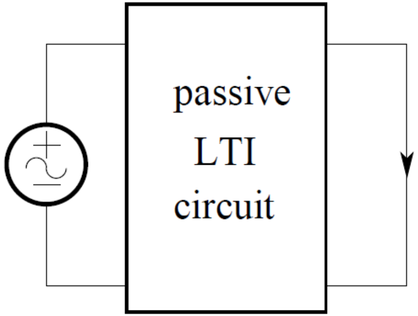
The additional imaginary part in phasor will not cause incorrectness because *imaginary input always leads to imaginary output in a linear system*. And by converting phasor back to waveform, the imaginary part is removed.

Input

$V_1\cos(\theta_1)$

$jV_1\cos(\theta_1)$

$e^{j\omega t}V_1 e^{j\theta_1}$



LTI: linear time-invariant

Output

$I_1\cos(\varphi_1)$

$jI_1\cos(\varphi_1)$

$e^{j\omega t}I_1 e^{j\varphi_1}$

The frequency part e^{jwt} is omitted but it will not change the results. Since this is a linear circuit, if the input is multiplied by this e^{jwt}, the output is also multiplied by it as this is a linear circuit. So we can simply add this time-variant characteristic after the phasor calculation.

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Topic 10-1-3: How to use phasors?

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10-1-3 HOW TO USE PHASORS?

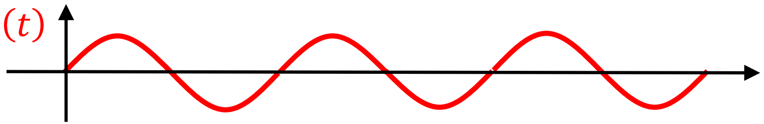
By using the phasor technique, the analysis is simplified.

Assume we have two waveforms

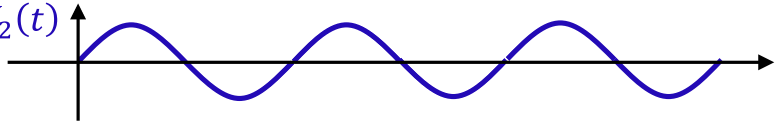
$$V_1(t) = V_{1m}\cos(\omega t + \theta_1)$$
$$V_2(t) = V_{2m}\cos(\omega t + \theta_2)$$

If they are in phase, i.e., $\theta_1 = \theta_2$

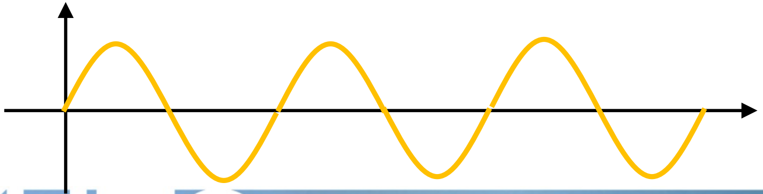
$V_1(t)$



$V_2(t)$

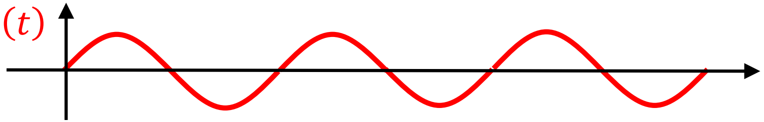


$V_1(t) + V_2(t) = (V_{m1} + V_{m2})\cos(\omega t + \theta_1)$

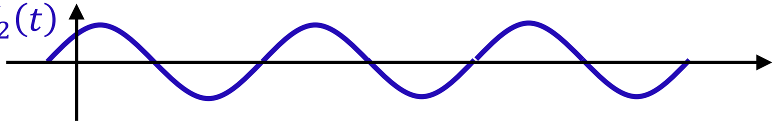


What if they are not in phase, i.e., $\theta_1 \neq \theta_2$

$V_1(t)$



$V_2(t)$



$V_1(t) + V_2(t) = ?$

By convert the waveforms (time-variant function) to phasors (complex numbers), the calculation is much easier.

10-1-3 HOW TO USE PHASORS?

Example: determine the waveform of $V_3(t) = V_1(t) + V_2(t)$

$$V_1(t) = 20\cos(\omega t - 45^0),\ V_2(t) = 10\cos(\omega t - 30^0)$$

Calculation procedure:

Step 1: convert waveform to phasor, make sure the waveform must be a cosine.

$$V_1(t) = 20\angle -45^0 = 20e^{-j45^0}$$

$$V_2(t) = 10\angle -30^0 = 10e^{-j30^0}$$

Either polar form or exponential form is ok

Step 2: convert polar form (exponential form) phasor to rectangular form because this form is easier to "add."

$$\begin{aligned} V_1(t) &= 20\cos(-45^0) + j20\sin(-45^0) \\ &= 20\cos(45^0) - j20\sin(45^0) \\ &= 20\times 0.707 - j20\times 0.707 = 14.14 - j14.14 \end{aligned}$$

$$\begin{aligned} V_2(t) &= 10\cos(-30^0) + j10\sin(-30^0) \\ &= 10\cos(30^0) - j10\sin(30^0) \\ &= 10\times 0.866 - j10\times 0.5 = 8.66 - j5 \end{aligned}$$

Step 3: Add the two phasors (complex numbers) in rectangular form.

$$\begin{aligned} V_1(t) + V_2(t) &= 14.14 - j14.14 + 8.66 - j5 \\ &= 22.8 - j19.4 \end{aligned}$$

Step 4: Convert the rectangular form phasor to polar/exponential form.

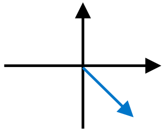
$$V_1(t) + V_2(t) = r\angle\theta = re^{\theta}$$

$$r = \sqrt{a^2 + b^2} = \sqrt{22.8^2 + 19.4^2} = 29.93$$

$$\theta = arctan\frac{b}{a} = arctan\frac{-19.4}{22.8} = -40.4^0?$$

Check whether the phase is correct using complex plane

Since a > 0, b < 0, the phasor locates in the 4th quadrant, the result -40.4^0 *is correct.*



Step 5: Convert the polar/exponential form phasor to waveform.

$$V_3(t) = 29.93\cos(\omega t - 40.4^0)$$

The frequency remains the same.

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Topic 10-2-1: Phasors for R, L and C

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10-2-1 PHASORS FOR R, L AND C

From time-domain to frequency-domain, **(AC sources)** sinusoidal waveforms coverts to phasors, and time-variant functions are replaced by complex numbers.

To analyse a circuit in frequency-domain, we also need to convert the components R, L and C into phasors. **The phasors for *R, L* and *C* are also be complex numbers.**

How to find the phasors for R, L, C?

The phasors for the components are determined by their V/I characteristics.

Phasor for R:

Assuming there is a voltage source of $V(t) = V_m \cos(\omega t + \theta)$ applied on a resistor R, find the current *I(t)*

V-I characteristic of R

$$I = \frac{V}{R}$$

$$R = \frac{V}{I}$$

Time domain:

$$V(t) = V_m \cos(\omega t + \theta)$$

$$I(t) = \frac{V(t)}{R} = \frac{V_m}{R}\cos(\omega t + \theta)$$

Frequency domain

$$\boldsymbol{V_R} = V_m e^{j\theta}$$

$$\boldsymbol{I_R} = \frac{V_m}{R} e^{j\theta}$$

$$\boxed{\boldsymbol{R} = \frac{\boldsymbol{V_R}}{\boldsymbol{I_R}} = Re^{j0} = R}$$

10-2-1 PHASORS FOR R, L AND C

How to find the phasors for R, L, C?

The phasors for the components are determined by their V/I characteristics.

Phasor for C:

Assuming there is a voltage source of $V(t) = V_m\, cos(\omega t + \theta)$ applied on a capacitor C, find the current $I(t)$

V-I characteristic of C

$$I(t) = C\frac{dV(t)}{dt}$$

$$V(t) = \frac{1}{C}\int_{t_0}^{t} i(t)dt + V(t_0)$$

Time domain:

$$V(t) = V_m\, cos(\omega t + \theta)$$

$$I(t) = C\frac{dV(t)}{dt}$$

$$= C\frac{d[V_m\cos(\omega t + \theta)]}{dt}$$

$$= CV_m(-\omega)\sin(\omega t + \theta)$$

$$= -\omega C V_m \cos(\omega t + \theta - 90^0)$$

$$= \omega C V_m \cos(\omega t + \theta - 90^0 + 180^0)$$

$$= \omega C V_m \cos(\omega t + (\theta + 90^0))$$

Frequency domain

$$\boldsymbol{V_C} = V_m e^{j\theta}$$

$$\boldsymbol{I_C} = \omega C V_m e^{j(\theta + 90^0)}$$

$$\boxed{\begin{aligned} \boldsymbol{C} = \frac{\boldsymbol{V_C}}{\boldsymbol{I_C}} &= \frac{1}{\omega C}e^{-j90^0} \\ &= -j\frac{1}{\omega C} \end{aligned}}$$

10-2-1 PHASORS FOR R, L AND C

How to find the phasors for R, L, C?

The phasors for the components are determined by their V/I characteristics.

Phasor for L:

Assuming there is a current of $I(t) = I_m\, cos(\omega t + \theta)$ flowing through an inductor of L

V-I characteristic of L

$$V(t) = L\frac{dI(t)}{dt}$$

$$I(t) = \frac{1}{L}\int_{t_0}^{t} V(t)\,dt + I(t_0)$$

Time domain:

$$I(t) = I_m\, cos(\omega t + \theta)$$

$$\begin{aligned} V(t) &= L\frac{dI(t)}{dt} \\ &= L\frac{d[I_m\, cos(\omega t + \theta)]}{dt} \\ &= LI_m(-\omega)\sin(\omega t + \theta) \\ &= -\omega LI_m \cos(\omega t + \theta - 90^0) \\ &= \omega LI_m \cos(\omega t + \theta - 90^0 + 180^0) \\ &= \omega LI_m \cos(\omega t + (\theta + 90^0)) \end{aligned}$$

Frequency domain

$$\boldsymbol{V_L} = \omega LI_m e^{j(\theta+90^0)}$$

$$\boldsymbol{I_L} = I_m e^{j\theta}$$

$$\boxed{\begin{aligned} \boldsymbol{L} = \frac{\boldsymbol{V_L}}{\boldsymbol{I_L}} &= \omega L e^{j90^0} \\ &= j\omega L \end{aligned}}$$

10-2-1 PHASORS FOR R, L AND C

Phasor for R, L, and C

$$\boxed{\boldsymbol{R} = \frac{\boldsymbol{V_R}}{\boldsymbol{I_R}} = Re^{j0} = R}$$

In frequency domain, resistor **R** is still R.

$$\boxed{\begin{aligned}\boldsymbol{L} = \frac{\boldsymbol{V_L}}{\boldsymbol{I_L}} &= \omega L e^{j90^0} \\ &= j\omega L\end{aligned}}$$

In frequency domain, inductor **L** is changes to $j\omega L$.

$$\boxed{\begin{aligned}\boldsymbol{C} = \frac{\boldsymbol{V_C}}{\boldsymbol{I_C}} &= \frac{1}{\omega C} e^{-j90^0} \\ &= -j\frac{1}{\omega C}\end{aligned}}$$

In frequency domain, capacitor **C** is changes to $-j\frac{1}{\omega C}$.

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Topic 10-2-2: Complex impedance

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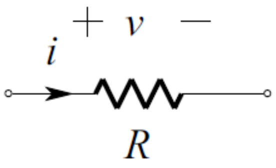
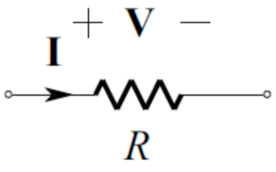
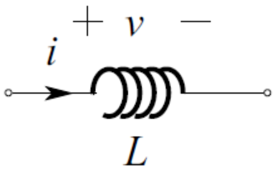
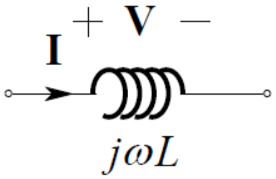
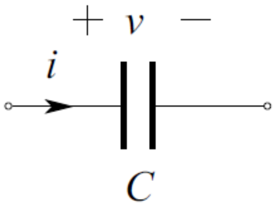
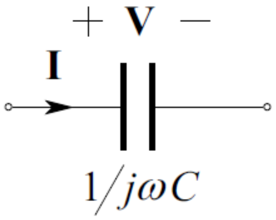
10-2-2 COMPLEX IMPEDANCE

Time-domain		*Frequency-domain*	
	$v = Ri$	$\mathbf{V} = R\mathbf{I}$	
	$v = L\dfrac{di}{dt}$	$\mathbf{V} = j\omega L\mathbf{I}$	
	$v = \dfrac{1}{C}\int i dt$	$\mathbf{V} = \dfrac{1}{j\omega C}\mathbf{I}$	

In frequency domain, all the three components have linear V-I characteristics.

The differential and integral operations are simplified to division and multiplication operations.

The phasor of inductor and capacitor behaves like a “imaginary resistor” in frequency domain.

10-2-2 COMPLEX IMPEDANCE

The phasors of R, L and C can be jointly called impedance. Impedance is a complex version of loads (resistance, capacitance and inductance). It describes a measure of opposition to AC voltages and currents. Impedances are complex and they have amplitude and phases. **Z = V / I**

- Real Part — Resistance R
- Imaginary Part — Reactance *j*X (capacitance or inductance)

$$Z = R + jX = \frac{V\angle\theta_v}{I\angle\theta_I} \text{ (Ohm)}$$

	For resistor R	For capacitor C	For inductance L
Retangular	$Z_R = R = R + j0$	$Z_C = -j\frac{1}{\omega C} = 0 - j\frac{1}{\omega C}$	$Z_L = j\omega L = 0 + j\omega L$
Polar	$R\angle 0^0$	$\frac{1}{\omega C}\angle -90^0$	$\omega L\angle 90^0$
Exponential	Re^{j0^0}	$\frac{1}{\omega C}e^{-j90^0}$	ωLe^{j90^0}

Impedances can be combined and manipulated like resistors except we use complex algebra.

All the rules apply for resistance in time domain apply for impedance in frequency domain.

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Topic 10-2-3: Example of complex impedance

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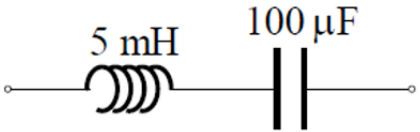
10-2-3 COMPLEX IMPEDANCE EXAMPLE

Example 1:

What's the impedance of this circuit when $\omega = 10^4$ and $\omega = 5000$

Series connection of L and C



$$Z_R = R$$
$$Z_L = j\omega L$$
$$Z_C = -j\frac{1}{\omega C}$$

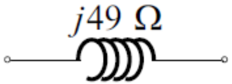
The impedance of capacitor and inductor is related to the angular frequency ω

When $\boldsymbol{\omega = 10^4}$,

$$Z_L = j\omega L = j(10^4 \times 5 \times 10^{-3})$$
$$= j50$$

$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{10^4 \times 100 \times 10^{-6}}$$
$$= -j1$$

$$Z_{eq} = Z_C + Z_L = j49$$



When $\boldsymbol{\omega = 5000}$,

$$Z_L = j\omega L = j(5000 \times 5 \times 10^{-3})$$
$$= j25$$

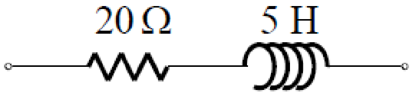
$$Z_C = -j\frac{1}{\omega C} = -j\frac{1}{5000 \times 100 \times 10^{-6}}$$
$$= -j2$$

$$Z_{eq} = Z_C + Z_L = j23$$

10-2-3 COMPLEX IMPEDANCE EXAMPLE

Example 2:

What's the impedance of this circuit **when $\boldsymbol{\omega} = 4$**



Series connection of R and C

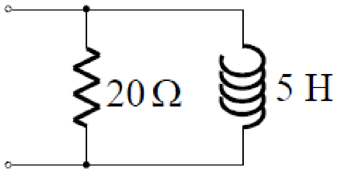
$$Z_R = 20$$

$$Z_L = j\omega L = j(4 \times 5) = j20$$

$$Z_{eq} = Z_R + Z_L = 20 + j20$$

Example 3:

What's the impedance of this circuit **when $\boldsymbol{\omega} = 4$**



Parallel connection of R and C

$$Z_R = 20$$

$$Z_L = j\omega L = j20$$

$$\frac{1}{Z_{eq}} = \frac{1}{Z_L} + \frac{1}{Z_R}$$

$$Z_{eq} = \frac{Z_R Z_L}{Z_R + Z_L} = \frac{20 \times j20}{20 + j20} = \frac{j20}{(1 + j)} = \frac{j20 \times (1 - j)}{(1 + j) \times (1 - j)}$$

$$= \frac{j20 + 20}{1 + 1} = 10 + j10$$

$$Z_R = R$$

$$Z_L = j\omega L$$

$$Z_C = -j\frac{1}{\omega C}$$