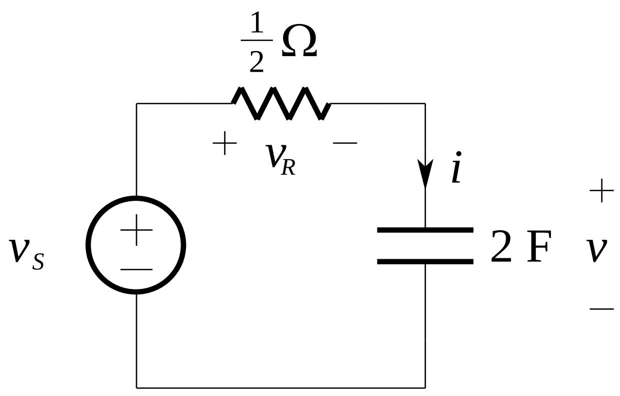
Capacitance

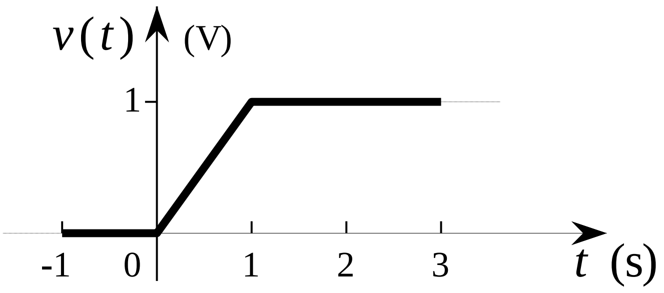
Worked Solutions

1.

Consider the circuit shown below:



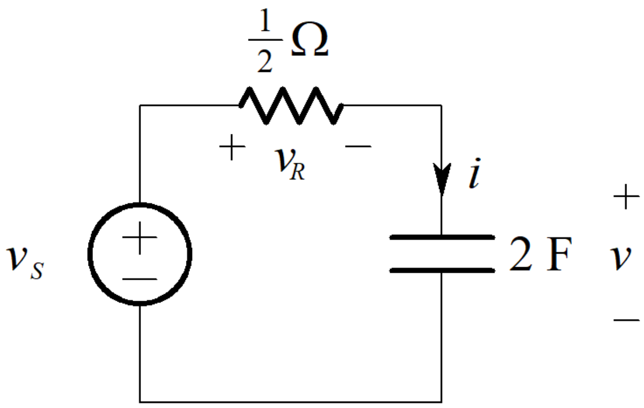
Suppose that the voltage $v(t)$ is:



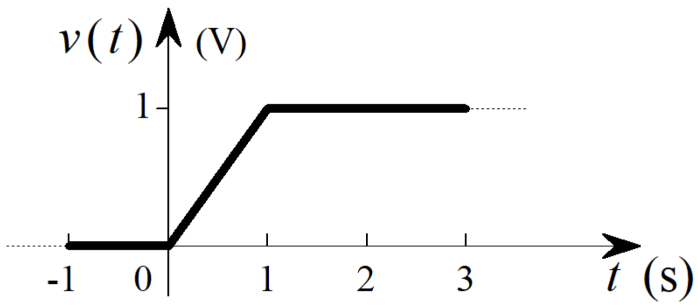
Find $i(t)$, $w_C(t)$, $p_R(t)$, $v_R(t)$ and $v_S(t)$, and sketch these functions.

1.

Consider the circuit shown below:



Suppose that the voltage $v(t)$ is:



Find $i(t)$, $w_C(t)$, $p_R(t)$, $v_R(t)$ and $v_S(t)$, and sketch these functions.

Solution

The function $v(t)$ can be separated into three parts: the constant voltage from $t = -1$ to $t = 0$, the linear function from $t = 0$ to $t = 1$, and the constant voltage from $t = 1$ to $t = 3$. For the linear function, it is increasing at a rate of $1Vs^{-1}$, which is equivalent to v(t) = 1t. Re-writing this information into a function:

$$v(t) = \begin{cases} 0\ V & for & -1 < t < 0 \\ t\ V & for & 0 < t < 1 \\ 1\ V & for & 1 < t < 3 \end{cases}$$

(a) $i(t)$

The formula for current through a capacitor is:

$$\boxed{i(t) = C\frac{dv}{dt}}$$

for $-1 < t < 0$,
v(t) is constant at 0 V, there is no rate of change therefore $\frac{dv}{dt} = 0$.

$$i(-1 < t < 0) = C\frac{dv}{dt}$$
$$i(-1 < t < 0) = 2 \times 0$$
$$i(-1 < t < 0) = 0\ A$$

for $0 < t < 1$,
v(t) is linearly increasing at a rate of 1 Vs^{-1}, or $\frac{dv}{dt}(1t) = 1$.

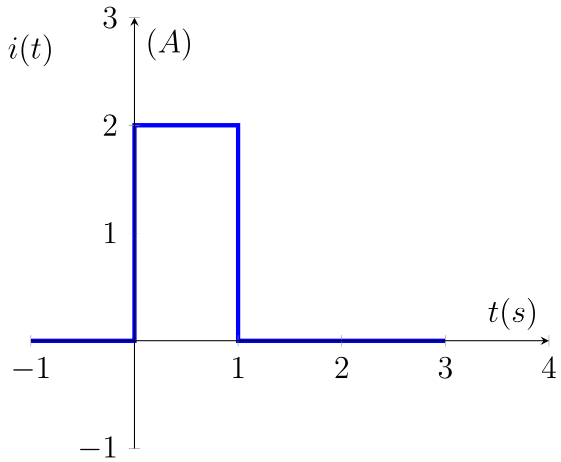
$$i(0 < t < 1) = C\frac{dv}{dt}$$
$$i(0 < t < 1) = 2 \times 1$$
$$i(0 < t < 1) = 2\ A$$

for $1 < t < 3$,
v(t) is constant at 1 V, there is no rate of change therefore $\frac{dv}{dt} = 0$.

$$i(1 < t < 3) = C\frac{dv}{dt}$$
$$i(1 < t < 3) = 2 \times 0$$
$$i(1 < t < 3) = 0\ A$$

$$\boxed{i(t) = \begin{cases} 0\ A & for \quad -1 < t < 0 \\ 2\ A & for \quad 0 < t < 1 \\ 0\ A & for \quad 1 < t < 3 \end{cases}}$$

Therefore, graphing the function i(t),



(b) $w_c(t)$

The formula for energy stored in a capacitor, w_c is:

$$\boxed{w_c = \frac{1}{2}Cv^2}$$

for $-1 < t < 0$,

$$\begin{aligned} w_c(-1 < t < 0) &= \frac{1}{2}(2)(0^2) \\ w_c(-1 < t < 0) &= 0 \ J \end{aligned}$$

for $0 < t < 1$,

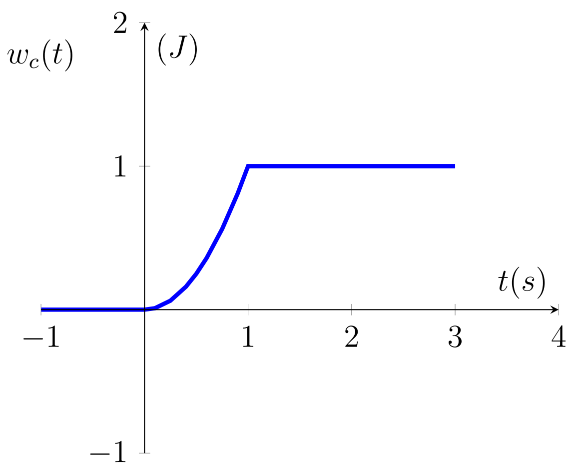
$$\begin{aligned} w_c(0 < t < 1) &= \frac{1}{2}(2)(t^2) \\ w_c(0 < t < 1) &= 1 \times t^2 \\ w_c(0 < t < 1) &= t^2 \ J \end{aligned}$$

for $1 < t < 3$,

$$\begin{aligned} w_c(1 < t < 3) &= \frac{1}{2}(2)(1^2) \\ w_c(1 < t < 3) &= 1 \ J \end{aligned}$$

$$\boxed{\mathrm{w}_c(t) = \begin{cases} 0 \ J & for \quad -1 < t < 0 \\ t^2 \ J & for \quad 0 < t < 1 \\ 1 \ J & for \quad 1 < t < 3 \end{cases}}$$

Therefore, graphing the function $w_c(t)$,



(c) $p_R(t)$

To find the power emitted from the resistor, we can use the formula:

$$\boxed{p = Ri^2}$$

as the capacitor and resistor are in series, therefore $i_c(t) = i_R(t)$.

for $-1 < t < 0$,

$$\begin{aligned} p_R(-1 < t < 0) &= R \times i(-1 < t < 0)^2 \\ p_R(-1 < t < 0) &= 0.5 \times 0 \\ p_R(-1 < t < 0) &= 0\ W \end{aligned}$$

for $0 < t < 1$,

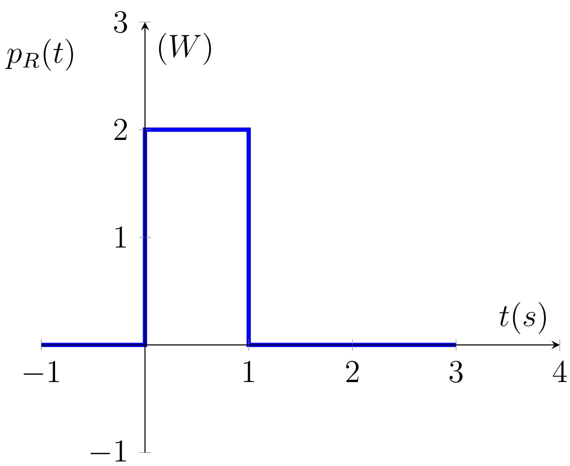
$$\begin{aligned} p_R(0 < t < 1) &= R \times i(0 < t < 1)^2 \\ p_R(0 < t < 1) &= 0.5 \times 2^2 \\ p_R(0 < t < 1) &= 2\ W \end{aligned}$$

for $1 < t < 3$,

$$\begin{aligned} p_R(1 < t < 3) &= R \times i(1 < t < 3)^2 \\ p_R(1 < t < 3) &= 0.5 \times 0 \\ p_R(1 < t < 3) &= 0\ W \end{aligned}$$

$$\boxed{\mathrm{p}_R(t) = \begin{cases} 0\ W & for \quad -1 < t < 0 \\ 2\ W & for \quad 0 < t < 1 \\ 0\ W & for \quad 1 < t < 3 \end{cases}}$$

Therefore, graphing the function $p_R(t)$



(d) $v_R(t)$

To find the voltage across the resistor, we can use Ohm's Law to solve for v.

$$\boxed{v = Ri}$$

for $-1 < t < 0$,

$$v_R(-1 < t < 0) = 0.5 \times i(-1 < t < 0)$$
$$v_R(-1 < t < 0) = 0.5 \times 0$$
$$v_R(-1 < t < 0) = 0\ V$$

for $0 < t < 1$,

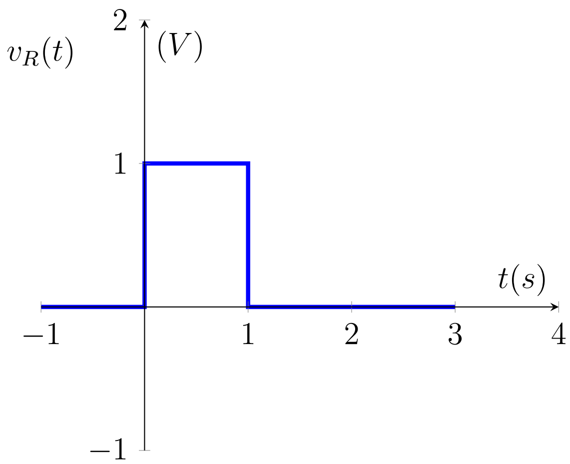
$$v_R(0 < t < 1) = 0.5 \times i(0 < t < 1)$$
$$v_R(0 < t < 1) = 0.5 \times 2$$
$$v_R(0 < t < 1) = 1\ V$$

for $1 < t < 3$,

$$v_R(1 < t < 3) = 0.5 \times i(1 < t < 3)$$
$$v_R(1 < t < 3) = 0.5 \times 0$$
$$v_R(1 < t < 3) = 0\ V$$

$$\boxed{\mathrm{v}_R(t) = \begin{cases} 0\ V & for \quad -1 < t < 0 \\ 1\ V & for \quad 0 < t < 1 \\ 0\ V & for \quad 1 < t < 3 \end{cases}}$$

Therefore, graphing the function $v_R(t)$



(e) $v_s(t)$

To solve for the voltage source, we can use KVL because we now know $v_c(t)$ and $v_R(t)$. Therefore:

$$\boxed{v_s(t) = v_c(t) + v_R(t)}$$

for $-1 < t < 0$,

$$\begin{aligned}
v_s(-1 < t < 0) &= v_c(-1 < t < 0) + v_R(-1 < t < 0) \\
v_s(-1 < t < 0) &= 0 + 0 \\
v_s(-1 < t < 0) &= 0\ V
\end{aligned}$$

for $0 < t < 1$,

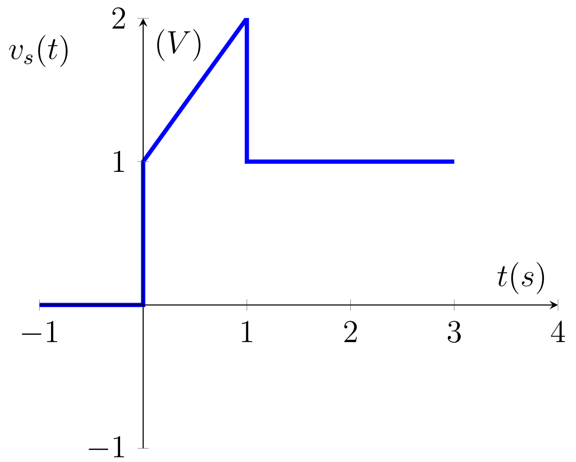
$$\begin{aligned}
v_s(0 < t < 1) &= v_c(0 < t < 1) + v_R(0 < t < 1) \\
v_s(0 < t < 1) &= t + 1 \\
v_s(0 < t < 1) &= (t + 1)\ V
\end{aligned}$$

for $1 < t < 3$,

$$\begin{aligned}
v_s(1 < t < 3) &= v_c(1 < t < 3) + v_R(1 < t < 3) \\
v_s(1 < t < 3) &= 1 + 0 \\
v_s(1 < t < 3) &= 1\ V
\end{aligned}$$

$$\boxed{\mathrm{v}_s(t) = \begin{cases} 0\ V & for \quad -1 < t < 0 \\ t + 1\ V & for \quad 0 < t < 1 \\ 1\ V & for \quad 1 < t < 3 \end{cases}}$$

Therefore, graphing the function $v_s(t)$



2.

The voltage $v(t)$ across a 2-F capacitor at time $t = 1\,\mathrm{s}$ is $\frac{1}{4}\,\mathrm{V}$. If the current through the capacitor is:

$$i(t) = \begin{cases} t & \text{for } 1 \leq t < 2 \\ 0 & \text{for } 2 \leq t < \infty \end{cases}$$

find $v(t)$ for $t \geq 1\,\mathrm{s}$.

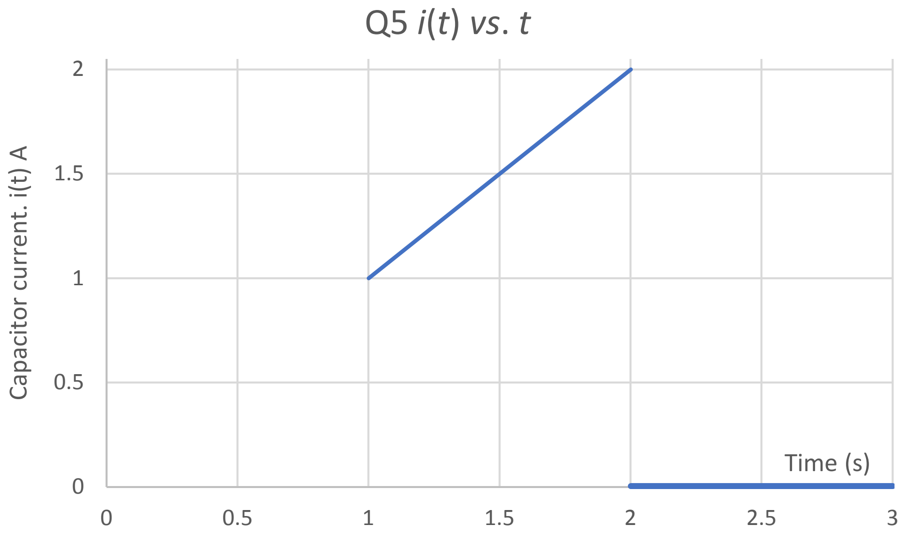
Capacitance

The voltage $v(t)$ across a 2 F capacitor at time t = 1 s is ¼ V. If the current through the capacitor is:

$$i(t) = \begin{cases} t & 1 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases}$$

Find $v(t)$ for $t \ge 1$ s.

Solution



The equation we need for this is:

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau + v(t_0) \qquad (1)$$

The first step is to find $v(t)$ for $1 \le t < 2$:

The time origin for the equation is at $t = 1$ so $v(t_0) = v(1\text{s}) =$ ¼ V, using equation 1

$$1 \le t < 2 \quad i(t) = t \qquad \Rightarrow 1 \le t < 2 \quad i(\tau) = \tau$$

$$v(t)=\frac{1}{2}\int_{1}^{t}\tau\,d\tau+\frac{1}{4}$$

$$v(t)=\frac{1}{2}\left[\frac{\tau^2}{2}\right]_1^t+\frac{1}{4}$$

$$v(t)=\frac{1}{2}\left[\frac{t^2-1}{2}\right]+\frac{1}{4}$$

$$v(t)=\frac{t^2-1}{4}+\frac{1}{4}$$

$$v(t)=\frac{t^2}{4}$$

Thus, for

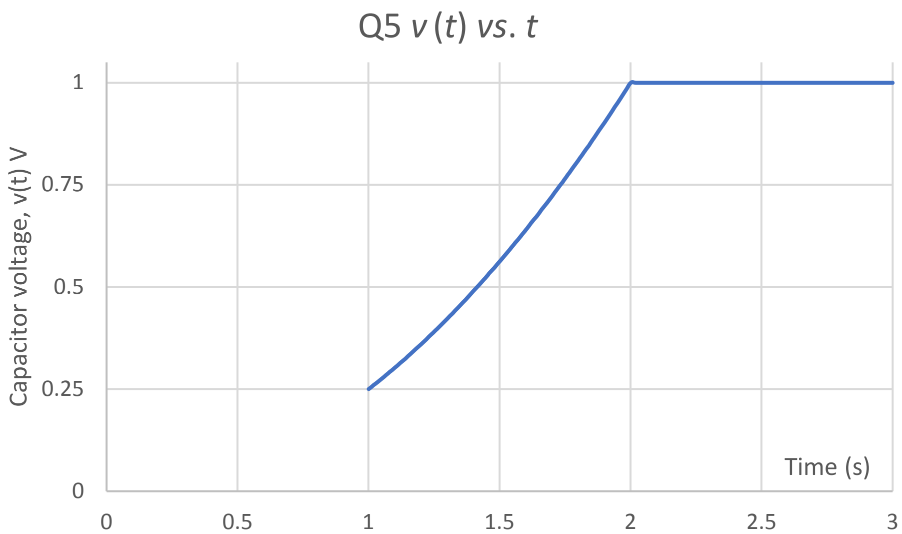
$$1\le t<2 \qquad v(t)=\frac{t^2}{4}$$

and

$$v(2)=1\text{ V}$$

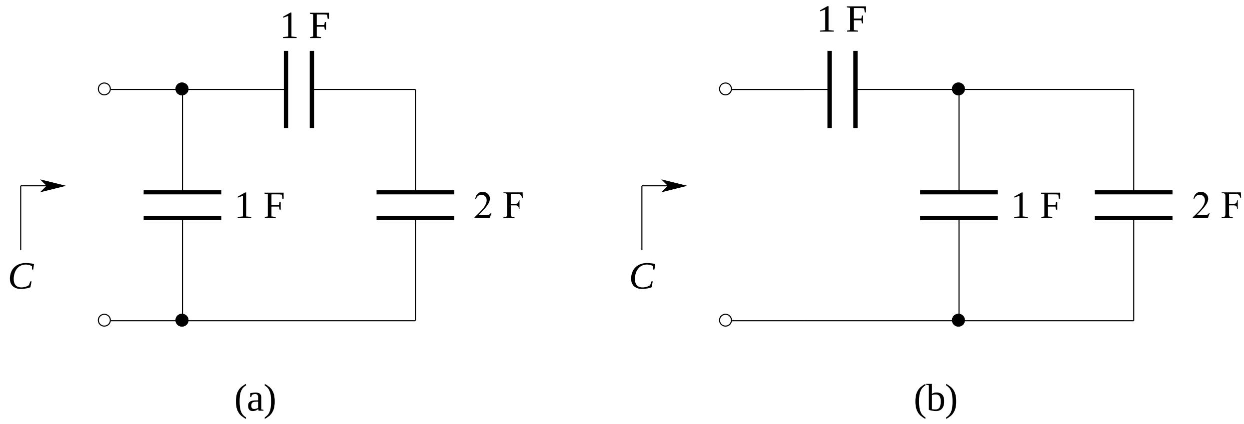
For $t\ge 2,\ i(t)=0$, thus no charge enters – or leaves - the capacitor and hence the voltage remains constant at 1 V.

$$v(t)=\begin{cases}\dfrac{t^2}{4} & 1\le t<2\\ 1 & 2\le t<\infty\end{cases}$$



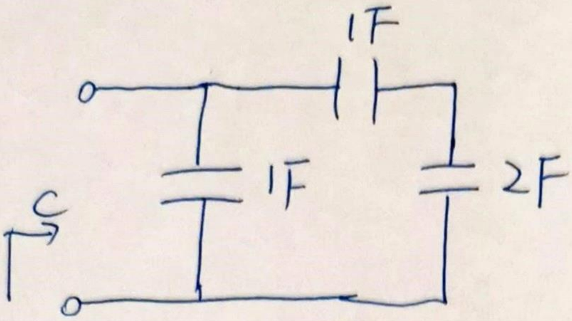
3.

For each of the circuits shown below, what is the value of C?



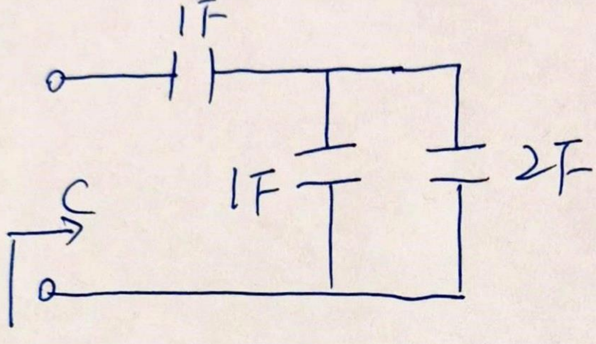
Solutions :

a.



$$C = 1F + \frac{1\times 2}{1+2}F = 1 + \frac{2}{3} = \frac{5}{3}F$$

b.



$$C = \frac{1F \times 3F}{(1+3)F} = \frac{3}{4}F.$$