# 37233 Linear Algebra

Subject coordinator and lecturer:

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CB 07.05.035

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## AUT 2024 classes

- Interactive lecture-style workshops: each Thursday 16:00
  (this is the main source of information and should be attended)
- Tutorials (2 hpw): each Friday as scheduled
  (including 30 min quick class tests in weeks 2-5, 7-8, and 10-11)
- Homework exercises: regular tasks for additional training
  (exercises and solutions published on Canvas)

Subject materials are published on Canvas

Please make sure you do receive e-mails on Canvas announcements

## Assessment scheme

- Quick class tests (30 min each, 8 tests at the tutorials): 30%

  — the first test 2 marks, the other seven tests 4 marks each

- Mid-term test (2 hours, at the tutorials on 26 April): 40%

  — it is required to gain at least 40% of the test mark

- Final test (2 hours, at the final tutorials on 17 May): 30%

  — it is required to gain at least 40% of the final test mark

# Important: next week is unusual



1. No on-campus lecture next week

Instead: recordings on Canvas to study online

(it covers the revision of essential material:
vector and matrix algebra, linear systems, row reduction,
homogeneous and inhomogeneous systems)

2. Tutorials (1 March) take place in room **CB 04.04.321**

Quick tests 1 are running next week on Friday (1 March)

(covering linear spaces and subspaces)

Linear algebra is a fundamental area of mathematics.

It is required almost everywhere where mathematics is involved — and not only where the analysis is explicitly linear:

- Science
  (physics, astronomy, chemistry, biology, statistics, ...)
- Engineering (mechanical, electrical, ...)
- Economics and business
- Transport, logistics, IT ...
- “Big Data” analysis, machine learning, AI, ...

(check the supplementary slides with a couple of examples)

# Pre-requisite knowledge required



- Mathematical logic and proofs
- Basic algebraic operations with vectors and matrices
- Matrix determinants
- Matrix inversion
- Eigenvectors and eigenvalues
- Solving linear systems / matrix row-reduction to echelon forms

There will be brief reminders on these matters as we go,
however these are relying on some pre-existing knowledge.

# Subject contents



- Linear spaces. Span. Linear (in)dependence.
- Basis. Coordinate systems.
- Linear transformations.
- Scalar product. Orthogonality. Projections.
- Least-squares approximation. Quadratic forms.
- Diagonalisation. Singular value decomposition.
- Free bonus topic: LU-decomposition / iterative methods.

## Fundamentals of linear algebra

- Linear spaces: definition and examples
- Subspaces: definition and examples

# Linear space: Definition

A *linear space* $V$ is a non-empty set of objects, for which two operations are defined so that $\forall\, \mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V$ and $\forall\, c,\ d \in \mathbb{R}$:

- addition $\quad (\mathbf{u} + \mathbf{v}) \in V$
- multiplication by scalars $\quad (c\,\mathbf{u}) \in V$

and these operations obey the following axioms:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\exists\, \mathbf{0} \quad : \quad \mathbf{u} + \mathbf{0} = \mathbf{u}$

(iv) $\exists\, (-\mathbf{u}) \quad : \quad \mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(v) $1 \cdot \mathbf{u} = \mathbf{u}$

(vi) $c\,(d\,\mathbf{u}) = (c\,d)\mathbf{u}$

(vii) $(c + d)\mathbf{u} = c\,\mathbf{u} + d\,\mathbf{u}$

(viii) $c\,(\mathbf{u} + \mathbf{v}) = c\,\mathbf{u} + c\,\mathbf{v}$

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(viii) $c\,(\mathbf{u} + \mathbf{v}) = c\,\mathbf{u} + c\,\mathbf{v}$

Consequences:

(ix) $\mathbf{0}$ is unique

(x) $(-\mathbf{u})$ is unique

(xi) $0 \cdot \mathbf{u} = \mathbf{0}$

(xii) $(-\mathbf{u}) = (-1) \cdot \mathbf{u}$

(xiii) $c \cdot \mathbf{0} = \mathbf{0}$

(xiv) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$

$(\mathbf{w} = \mathbf{u} - \mathbf{v}$ if $\mathbf{w} + \mathbf{v} = \mathbf{u})$

For example, prove the property that: $0 \cdot \mathbf{u} = \mathbf{0}$

Consider $0 \cdot \mathbf{u} + \mathbf{u}$, and apply the axioms as follows:

$$0 \cdot \mathbf{u} + \mathbf{u} \overset{\alpha\xi\, 5}{=} 0 \cdot \mathbf{u} + 1 \cdot \mathbf{u} \overset{\alpha\xi\, 7}{=} (0+1) \cdot \mathbf{u} = 1 \cdot \mathbf{u} \overset{\alpha\xi\, 5}{=} \mathbf{u}$$

Now add $-\mathbf{u}$ to each side of the obtained equation:

$$\big(0 \cdot \mathbf{u} + \mathbf{u}\big) + (-\mathbf{u}) = \mathbf{u} + (-\mathbf{u}) \quad \overset{\alpha\xi\, 2}{\Longrightarrow}$$

$$0 \cdot \mathbf{u} + \big(\mathbf{u} + (-\mathbf{u})\big) = \mathbf{u} + (-\mathbf{u}) \quad \overset{\alpha\xi\, 4}{\Longrightarrow}$$

$$0 \cdot \mathbf{u} + \mathbf{0} = \mathbf{0} \quad \overset{\alpha\xi\, 3}{\Longrightarrow} \quad 0 \cdot \mathbf{u} = \mathbf{0} \quad \blacksquare$$

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Consequences:

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(xiv) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$

**1.** Vector space

Consider vectors with $n$ components (real numbers).

Vector is an ordered list of numbers. For example $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

Multiplication by a scalar: $c \cdot \mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$

Addition:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} + \mathbf{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}$$

Vector with all entries equal to $0$ is called a *zero vector* $\mathbf{0}$

**1.** Vector space $\mathbb{R}^n$ is a linear space.

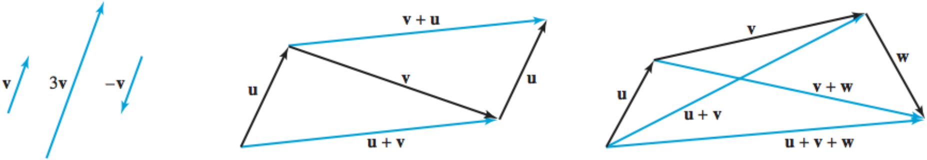
Consider vectors with $n$ components (real numbers). Addition and multiplication by scalars results in a vector with $n$ components.

The collection of all such vectors forms a *vector space* $\mathbb{R}^n$.

Properties of a vector space satisfy the axioms of linear space.

**2.** A set of all arrows (directed line segments) on a plane, with

- multiplication $c\,\mathbf{v}$ defines an arrow $|c|$ times the length of $\mathbf{v}$, pointing in the same ($c > 0$) or opposite ($c < 0$) direction
- addition defined by parallelogram rule



**3.** Let $\mathbb{S}$ be a space of "double-infinite" sequences of numbers:

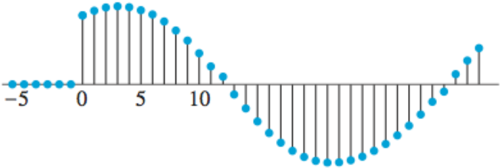
$$\mathfrak{Y} = \langle y_k \rangle = \{\dots\ y_{-2},\ y_{-1},\ y_0,\ y_1,\ y_2,\dots\}$$

If $\mathfrak{Z} = \langle z_k \rangle$ is another element of $\mathbb{S}$, the sum is $\mathfrak{Y} + \mathfrak{Z} = \langle y_k + z_k \rangle$.

The scalar multiple is defined by $c \cdot \mathfrak{Y} = \langle c y_k \rangle$.

For $\mathbb{S}$, all the axioms can be verified, so this is a linear space.

Similar linear spaces arise in engineering when a signal (such as electrical, optical or mechanical) is measured at discrete times.



**4.** For $n \geqslant 0$ let $\mathbb{P}_n$ be a set of polynomials of a degree up to $n$

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

where variable $t$ and coefficients $a_0 \ldots a_n$ are real numbers.

The degree of $\mathbf{p}$ is the highest power of $t$ with non-zero coefficient (for $\mathbf{p}(t) = a_0 \neq 0$ it is zero). The *zero polynomial* has all $a_i = 0$.

Given $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2 + \ldots + b_n t^n$, the sum is defined as

$$\{\mathbf{p}+\mathbf{q}\}(t) = \mathbf{p}(t)+\mathbf{q}(t) = (a_0+b_0)+(a_1+b_1)t+\ldots+(a_n+b_n)t^n$$

The scalar multiple $c\,\mathbf{p}$ is the polynomial defined by

$$c\,\mathbf{p}(t) = ca_0 + (ca_1)t + (ca_2)t^2 + \ldots + (ca_n)t^n$$

The axioms of a vector space are satisfied so $\mathbb{P}_n$ is a linear space.

**5.** Let $\mathbb{F}$ be a space of all real-valued functions defined on a number space $\mathbb{R}$.

- Addition is defined as the function $\{\mathbf{f}+\mathbf{g}\}$ with the value equal to $\mathbf{f}(t)+\mathbf{g}(t)\ \forall\, t \in \mathbb{R}$
- Scalar multiplication by $c$ is defined as the function $c\,\mathbf{f}$ with the value $c \cdot \mathbf{f}(t)\ \forall\, t \in \mathbb{R}$
- Two functions are equal if their values are equal $\forall\, t \in \mathbb{R}$
- The zero element in $\mathbb{F}$ is $\mathbf{f}_0(t) \equiv 0\ \forall\, t \in \mathbb{R}$
- The negative of $\mathbf{f}$ is $\bar{\mathbf{f}}$ such that $\bar{\mathbf{f}}(t) = -\mathbf{f}(t)$

All the axioms are valid for $\mathbb{F}$ so it is a linear space.

**6.** A set $\mathbb{M}_2^3$ of all $2 \times 3$ matrices of real numbers

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \end{bmatrix}$$

with the addition defined as the usual matrix addition, and multiplication by a scalar defined as the usual multiplication of a matrix by a scalar, — is a linear space.

All the axioms of the linear space are satisfied for this set.

Zero element in this linear space is the matrix of all zeros.

An example of operation with such elements:

$$-3 \cdot \begin{bmatrix} 0 & 1 & -2 \\ 3 & -4 & 5 \end{bmatrix} + 2 \cdot \begin{bmatrix} -9 & 8 & 0 \\ 7 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -18 & 13 & 6 \\ 5 & 12 & -3 \end{bmatrix}$$

More generally, consider a linear space of $m \times n$ matrices.

**7.** The set of all positive numbers with the usual addition and multiplication is not a linear space: there is no zero element, and no opposite element to satisfy the axioms (iii) and (iv).

**8.** Consider as set of polynomials of a degree *equal* to $n = 3$

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \qquad \text{for} \quad t \in \mathbb{R}$$

where coefficients $a_0 \ldots a_3$ are real numbers, $a_3 \neq 0$, with the usual operations of summation and multiplication by scalars.

Take, for example, the sum of these two third-degree polynomials:

$$\mathbf{p}_1(t) = x^3 + x^2 - 3x + 1 \qquad \text{and} \qquad \mathbf{p}_2(t) = -x^3 + 2x - 2$$

$$\mathbf{p}_1(t) + \mathbf{p}_2(t) = x^2 - x - 1$$

The result is not a third-degree polynomial.

Thus, definition is violated and the above set is not a linear space.

**9.** Consider all positive numbers: define "addition" as arithmetic multiplication, and "multiplication by a scalar" as taking an element to the power of that scalar.

For any positive numbers $x,\, y,\, z > 0$, and any $\alpha,\, \beta \in \mathbb{R}$:

$$x \oplus y \equiv x \cdot y > 0 \qquad \text{and} \qquad \alpha \odot x \equiv x^{\alpha} > 0$$

1. $x \oplus y \equiv x \cdot y = y \cdot x \equiv y \oplus x$
2. $(x \oplus y) \oplus z \equiv (x \cdot y) \cdot z = x \cdot (y \cdot z) \equiv x \oplus (y \oplus z)$
3. There is a "zero element" $\emptyset \equiv 1$ : $\quad x \oplus \emptyset \equiv x \cdot 1 = x$
4. There is an "opposite element" $\bar{x} \equiv \dfrac{1}{x}$ : $\quad x \oplus \bar{x} \equiv x \cdot \dfrac{1}{x} = 1 \equiv \emptyset$
5. $1 \odot x \equiv x^1 = x$
6. $\alpha \odot (\beta \odot x) \equiv (x^{\beta})^{\alpha} = x^{\beta\alpha} = x^{\alpha\beta} \equiv (\alpha\, \beta) \odot x$
7. $(\alpha + \beta) \odot x \equiv x^{(\alpha+\beta)} = x^{\alpha} \cdot x^{\beta} \equiv (\alpha \odot x) \oplus (\beta \odot x)$
8. $\alpha \odot (x \oplus y) \equiv (x \cdot y)^{\alpha} = x^{\alpha} \cdot y^{\alpha} \equiv (\alpha \odot x) \oplus (\alpha \odot y)$

All the axioms are satisfied so for these operations this is a linear space!

# Subspaces

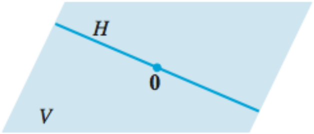
**Definition:** A **subspace** $H$ of a linear space $V$ is a subset of elements with the following properties:

- $H$ is closed under addition: $\forall\,(\mathbf{u},\mathbf{v}) \in H,\ \ (\mathbf{u}+\mathbf{v}) \in H$
- $H$ is closed under multiplication by scalars:
  $\forall\,\mathbf{u} \in H$ and $\forall\,c \in \mathbb{R},\ c\,\mathbf{u} \in H$

Every subspace is a linear space and satisfies the axioms.

Property of any subspace: $H$ includes the zero element of $V$

(proof: take $c = 0$, then: $0 \cdot \mathbf{v} \in H$, but $0 \cdot \mathbf{v} = \mathbf{0} \quad \Rightarrow \quad \mathbf{0} \in H$)

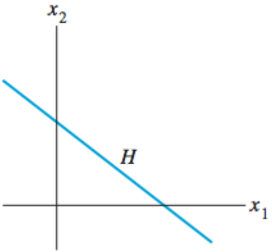


1. A set consisting of only the zero element of a linear space $V$ is a subspace of $V$ and is called *zero subspace* $\{\mathbf{0}\}$.

2. Consider space $\mathbb{P}$ of all polynomials with real coefficients, with operations in $\mathbb{P}$ defined as for real-valued functions. Then $\mathbb{P}$ is a subspace of the space $\mathcal{F}$ of all real-valued functions operating on $\mathbb{R}$, and $\mathbb{P}_n$ is the subspace of $\mathbb{P}$.

3. A line within $\mathbb{R}^2$, not passing through the origin, is not a subspace of $\mathbb{R}^2$ — it does not contain the $\mathbf{0}$ vector of $\mathbb{R}^2$.

4. A plane within $\mathbb{R}^3$, not including the origin, is not a subspace of $\mathbb{R}^3$ — this plane does not contain the $\mathbf{0}$ vector of $\mathbb{R}^3$.

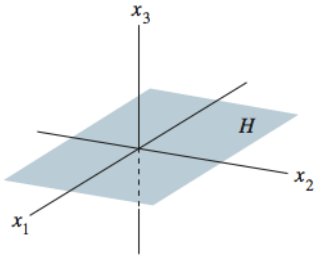


5. The entire vector space $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$. Vectors in $\mathbb{R}^3$ have three entries whereas vectors in $\mathbb{R}^2$ have two. However, a set like

$$H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} \right\}, \quad (s, t) \in \mathbb{R}$$

is a subset of $\mathbb{R}^3$ that "looks" exactly like $\mathbb{R}^2$.

Indeed, this subset is closed:
Any multiplication by a scalar or any addition of two vectors, produces a vector from this subset (because the third component is always zero).



6. Check if the set of all functions $f(t) = a \cdot 2^{(t+2)}$ is a subspace of the linear space of $g(t) = b \cdot 2^t + c \cdot 3^t$ where $a,\ b,\ c,\ t \in \mathbb{R}$.

Any $f(t)$ belongs to $\{g(t)\}$, whereby $b = 4a$ and $c = 0$.

Consider a sum of any two functions like $f(t)$:

$$f_1(t) + f_2(t) = a_1 \cdot 2^{(t+2)} + a_2 \cdot 2^{(t+2)} = (a_1 + a_2) \cdot 2^{(t+2)}$$

which belongs to $\{f(t)\}$ (with $a = a_1 + a_2$).

Considering any $f(t)$ multiplied by a scalar:

$$d \cdot f(t) = d \cdot (a \cdot 2^{(t+2)}) = (da) \cdot 2^{(t+2)}$$

we see the result belongs to $\{f(t)\}$ (with $a \rightarrowtail da$).

Thus, the requirements for being a subspace are fulfilled.

7. Check if the set of all polynomials such that $\mathbf{p}(0) = 1$, is a subspace of the linear space of all polynomials.

Any polynomial $p(0) = 1$ certainly belongs to the space $\mathbb{P}$.

Consider the sum of any two such polynomials at point $t = 0$:

$$\mathbf{p}_1(0) + \mathbf{p}_2(0) = 1 + 1 = 2$$

which is not a polynomial that equals to 1 for $t = 0$.

This demonstrates that the set is not closed under addition — therefore it is not a subspace of $\mathbb{P}$.

Needless to say, it is also not closed under multiplication.

# Questions on these topics?

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(covering linear spaces and subspaces)

3. Next lecture on campus: 7 March