University of Technology Sydney
School of Mathematical and Physical Sciences

37233 Linear Algebra

Tutorial 1 — solution guide

Question 1

One of the possible proofs takes the benefit of a property derived in the lecture:

$$0 \cdot \mathbf{u} = \mathbf{0}$$

Based on this property, we can write

$$x \cdot \mathbf{0} \overset{(above)}{=} x \cdot (0 \cdot \mathbf{u}) \overset{\alpha\xi\,6}{=} (x \cdot 0) \cdot \mathbf{u} = 0 \cdot \mathbf{u} \overset{(above)}{=} \mathbf{0}$$

(NB: the axiom numbers refer to the order provided with the lectures)

Question 2

(a) If a vector located in the first quadrant is multiplied by a negative number, this results in a vector located in the third quadrant. Thus, this set is not closed under multiplication by a scalar. Therefore, this set is not a linear space. Note that, needless to say, axiom (iv) is violated either.

(b) All the axioms are satisfied for these functions, in the same way as for numbers with the corresponding arithmetic operations. Therefore this set is a linear space.

Question 3

(a) Any solution $\mathbf{x}$ to this equation is a vector with n components, so $\mathbf{x} \in \mathbb{R}^n$. Suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions to the system. Then $\mathbf{x}_1 + \mathbf{x}_2$ is also a solution, because

$$\mathbf{A}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{A}\mathbf{x}_1 + \mathbf{A}\mathbf{x}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

Also $c \cdot \mathbf{x}_1$ is a solution, because

$$\mathbf{A}(c \cdot \mathbf{x}_1) = c \cdot \mathbf{A}\mathbf{x}_1 = c \cdot \mathbf{0} = \mathbf{0}$$

Thus the conditions for a subspace are fulfilled: $(\mathbf{x}_1 + \mathbf{x}_2) \in X$ and $(c \cdot \mathbf{x}_1) \in X$, therefore X is a subspace of $\mathbb{R}^n$. And if there is only has the trivial solution $(\mathbf{0})$, then it is the zero subspace of $\mathbb{R}^n$.

(b) Suppose Γ is a subspace of V. Then $\mathbf{0} \in \Gamma$, so it can be represented as $\mathbf{0} = \mathbf{v}_0 + \mathbf{x}_0$ where $\mathbf{x}_0 \in H$. By the axioms of a linear space, $-\mathbf{x}_0 + \mathbf{x}_0 = \mathbf{0}$, and then $\mathbf{v}_0 = -\mathbf{x}_0$ and thereby $\mathbf{v}_0 \in H$. However this is a contradiction to the definition of Γ. Therefore, Γ is not a subspace of V.

Question 4

Following the same steps as shown in the lecture, we prove that $0 \cdot \mathbf{u} = \mathbf{0}$ Based on this, we can write

$$(-1) \cdot \mathbf{u} + \mathbf{u} \overset{\alpha\xi\,5}{=} (-1) \cdot \mathbf{u} + 1 \cdot \mathbf{u} \overset{\alpha\xi\,7}{=} (-1+1) \cdot \mathbf{u} = 0 \cdot \mathbf{u} \overset{(above)}{=} \mathbf{0}$$

Now, once again, add $-\mathbf{u}$ to each side of the obtained equation:

$$\begin{aligned}
\big((-1) \cdot \mathbf{u} + \mathbf{u}\big) + (-\mathbf{u}) &= \mathbf{0} + (-\mathbf{u}) && \overset{\alpha\xi\,2}{\Longrightarrow} \\
(-1) \cdot \mathbf{u} + \big(\mathbf{u} + (-\mathbf{u})\big) &= \mathbf{0} + (-\mathbf{u}) && \overset{\alpha\xi\,4}{\Longrightarrow} \\
(-1) \cdot \mathbf{u} + \mathbf{0} &= \mathbf{0} + (-\mathbf{u}) && \overset{\alpha\xi\,1}{\Longrightarrow} \\
(-1) \cdot \mathbf{u} + \mathbf{0} &= (-\mathbf{u}) + \mathbf{0} && \overset{\alpha\xi\,3}{\Longrightarrow} \\
(-1) \cdot \mathbf{u} &= (-\mathbf{u}) && \blacksquare
\end{aligned}$$

Question 5

Let us check if all the axioms are fulfilled with the defined operations. For any negative numbers x, $y < 0$, the "sum" $x \oplus y \equiv x \cdot y > 0$, so does not belong to negative numbers. Thereby this set is not closed under "addition" as defined, and is not a linear space.

Alternatively, we can point out that $\alpha \odot x \equiv x^\alpha$ is only negative for an odd integer α, but it is positive for an even integer α, and it is a complex number for non-integer α. Thereby this set is not closed under "multiplication by scalar" as defined, and is not a linear space.

Question 6

(a) As such, any cubic polynomial belongs to the space $\mathbb{P}_4$. However, let us consider, for example, the following two cubic polynomials:

$$\mathbf{p}_1(t) = t^3 + 2t^2 + 3t + 4 \qquad \text{and} \qquad \mathbf{p}_2(t) = -t^3 + t + 2$$

Taking the sum of these two polynomials yields $\mathbf{p}_1(t) + \mathbf{p}_2(t) = 2t^2 + 4t + 6$ which is not a cubic polynomial. So, as shown with this example, the set of cubic polynomials is not closed under addition, and therefore it is not a subspace of $\mathbb{P}_4$.

(b) As such, any function like $f(t)$ belongs to the linear space of functions $g(t)$, whereby this is just a particular case with $b = 0$. Consider a sum of any two functions like $f(t)$:

$$f_1(t) + f_2(t) = a_1 \cos t + a_2 \cos t = (a_1 + a_2) \cos t$$

which is also a function from the set of $f(t)$.

Considering any $f(t)$ multiplied by a scalar, we see the result belongs to the set of $f(t)$:

$$d \cdot f(t) = d \cdot (a \cos t) = (d \cdot a) \cos t$$

Thus, the requirements for being a subspace are fulfilled, so the set $\mathcal{F}$ of functions $f(t)$ is a subspace of the space $\mathcal{G}$ of functions $g(t)$.