University of Technology Sydney

School of Mathematical and Physical Sciences

37233 Linear Algebra

Solutions 10

Question 1

(a) The characteristic equation for

$$\mathbf{A}^\mathsf{T}\mathbf{A} = \begin{bmatrix} 9 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

follows as: $(9-\lambda)(4-\lambda)(1-\lambda) - 3\cdot(4-\lambda)\cdot 3 = -\lambda^3 + 14\lambda^2 - 40\lambda = 0$
from where the roots are $\lambda_1 = 10,\ \lambda_2 = 4$ and $\lambda_3 = 0,$
so the singular values of $\mathbf{A}$ are $\sigma_1 = \sqrt{10},\ \sigma_2 = 2,\ \sigma_3 = 0,$ and

$$\Sigma = \begin{bmatrix} \sqrt{10} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

The eigenvectors are then found from solving $(\mathbf{A} - \lambda_i\mathbf{I}) = \mathbf{0}$ equations:

$$\begin{bmatrix} -1 & 0 & 3 \\ 0 & -6 & 0 \\ 3 & 0 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 0 & 3 \\ 0 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_3 = \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}$$

Forming an orthonormal set $\mathbf{u}_1,\ \mathbf{u}_3,\ \mathbf{u}_3$ from this orthogonal basis yields

$$\mathbf{U} = \begin{bmatrix} 3/\sqrt{10} & 0 & -1/\sqrt{10} \\ 0 & 1 & 0 \\ 1/\sqrt{10} & 0 & 3/\sqrt{10} \end{bmatrix}$$

Finally, the $\mathbf{W}$ matrix can be formed from

$$\mathbf{w}_1 = \frac{\mathbf{A}\mathbf{u}_1}{\sigma_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w}_2 = \frac{\mathbf{A}\mathbf{u}_2}{\sigma_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Thus

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3/\sqrt{10} & 0 & 1/\sqrt{10} \\ 0 & 1 & 0 \\ -1/\sqrt{10} & 0 & 3/\sqrt{10} \end{bmatrix}$$

(b) Given the above findings $\mathbf{A}$ can be decomposed as

$$\mathbf{A} = \sqrt{10}\begin{bmatrix} 3/\sqrt{10} & 0 & 1/\sqrt{10} \\ 0 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

which is quite clear; note the rank is 1 for each matrix.

Question 2

The matrix of this quadratic form is

$$\mathbf{A} = \begin{bmatrix} 9 & -4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) The characteristic equation for $\mathbf{A}$ follows as

$$\Big((9-\lambda)(3-\lambda)-16\Big)(3-\lambda) = \big(\lambda^2 - 12\lambda + 11\big)(3-\lambda) = 0$$

from where the roots are $\lambda_1 = 11$, $\lambda_2 = 3$ and $\lambda_3 = 1$ (the descending order is not compulsory but any selected order is to be kept throughout), and the corresponding unit eigenvectors follow as

$$\begin{bmatrix} -2 & -4 & 0 \\ -4 & -8 & 0 \\ 0 & 0 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{u}_1 = \begin{bmatrix} -\dfrac{2}{\sqrt{5}} \\ \dfrac{1}{\sqrt{5}} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{u}_2$$

$$\begin{bmatrix} 8 & -4 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \mathbf{v}_3 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{u}_3 = \begin{bmatrix} \dfrac{1}{\sqrt{5}} \\ \dfrac{2}{\sqrt{5}} \\ 0 \end{bmatrix}$$

whereby $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are the sought principal axes.

(b) The corresponding change of variables is then described by

$$\mathbf{x} = \begin{bmatrix} -2/\sqrt{5} & 0 & 1/\sqrt{5} \\ 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix} \mathbf{y} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{bmatrix} \mathbf{x}$$

(c) In terms of $\mathbf{y}$ variables, the matrix of the quadratic form is diagonal:

$$\mathbf{D} = \begin{bmatrix} 11 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: depending on the selected order of the eigenvalues, the order of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ as the columns of the change of variable matrix will be different accordingly; however that order is in strict correspondence to the order of values along the diagonal of $\mathbf{D}$.