University of Technology Sydney
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37233 Linear Algebra

Tutorial 10 — solutions guide

Question 1

The characteristic equation is $(1-\lambda)^2 - 25 = 0$ so the eigenvalues are $\lambda_1 = 6,\ \lambda_2 = -4$. Then the (orthogonal as the eigenvalues are different) eigenvectors follow as

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Thus

$$\mathbf{A} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

The spectral decomposition then follows as

$$\mathbf{A} = 6\mathbf{u}_1\mathbf{u}_1^\top - 4\mathbf{u}_2\mathbf{u}_2^\top = 6\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} - 4\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = 3\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Question 2

(a): $\mathbf{A} = \begin{bmatrix} 8 & -3 & 2 \\ -3 & 7 & -1 \\ 2 & -1 & -3 \end{bmatrix}$ (b): $\mathbf{A} = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 0 & -4 \\ 3 & -4 & 0 \end{bmatrix}$

Question 3

The matrix for the quadratic form is $\mathbf{A} = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix}$.

From the characteristic equation $(3-\lambda)(6-\lambda) - 4 = \lambda^2 - 9\lambda + 14 = 0$ the eigenvalues follow as $\lambda_1 = 7$ and $\lambda_2 = 2$ and so then for the eigenvectors

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

The matrices for $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^\top$ decomposition are therefore

$$\mathbf{P} = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$$

whereby the change of variables is $\mathbf{x} = \mathbf{P}\mathbf{y}$ and $\mathbf{y} = \mathbf{P}^\top\mathbf{x}$, and the new quadratic form is

$$Q(y) = 7y_1^2 + 2y_2^2.$$

Question 4

Thanks to the orthogonality of the columns, $\mathbf{A}^\mathsf{T}\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and the eigenvalues are, placing them in descending order, $\lambda_1 = 3$ and $\lambda_2 = 2$; the corresponding eigenvectors are then $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, which are orthonormal already.

(a) The singular values are then $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{2}$.

(b) The maximum length for $\mathbf{Ax}$ is achieved with the eigenvector corresponding to the largest singular value, which is $\mathbf{u}_1$; then $\mathbf{A}\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\|\mathbf{A}\mathbf{u}_1\| = \sqrt{3}$.

(c) We already have:

$$\mathbf{U} = \begin{bmatrix} \mathbf{u}_1 \, \mathbf{u}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad \Sigma = \begin{bmatrix} \mathbf{D} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

For $\mathbf{W}$, we easily obtain two of the vectors:

$$\mathbf{w}_1 = \frac{1}{\sigma_1}\mathbf{A}\mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{w}_2 = \frac{1}{\sigma_2}\mathbf{A}\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

To form the third required vector, we need to take a linearly independent vector and orthogonalise it with a Gram-Schmidt step.

Let us take $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ so then $\mathbf{w}_3 = \mathbf{v}_3 - \dfrac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \dfrac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$

and so finally $\mathbf{W} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$.

The resulting SVD is

$$\mathbf{A} = \mathbf{W}\Sigma\mathbf{U}^\top = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(after the workshop, verify this makes a correct decomposition).