University of Technology Sydney
School of Mathematical and Physical Sciences

37233 Linear Algebra

Solutions 11

Question 1

The matrix of the system is $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -3 & 0 \\ 1 & -1 \end{bmatrix}$ and the right-hand side is $\mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}$.

Please check that the system as such is indeed inconsistent (e.g. by row reduction).

Then $\mathbf{A}^\mathsf{T}\mathbf{A} = \begin{bmatrix} 15 & 0 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{A}^\mathsf{T}\mathbf{b} = \begin{bmatrix} 10 \\ -7 \end{bmatrix}$

so by row reduction: $\left[\begin{array}{cc|c} 15 & 0 & 10 \\ 0 & 3 & -7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2/3 \\ 0 & 1 & -7/3 \end{array}\right] \quad \Rightarrow \quad \check{\mathbf{x}} = \begin{bmatrix} 2/3 \\ -7/3 \end{bmatrix}$

and we can find the error of this approximation

$$\mathbf{b} - \mathbf{A}\check{\mathbf{x}} = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ -3 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \\ 0 \end{bmatrix} \qquad \text{so} \qquad \|\mathbf{b} - \mathbf{A}\check{\mathbf{x}}\| = 4\sqrt{3}$$

Question 2

(a) Denoting the columns of $\mathbf{A}$ as $\mathbf{a}_i$, apply the Gram-Schmidt process

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \mathbf{a}_3 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{a}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

(b) For QR decomposition we need to normalise $\mathbf{v}_i$; they have all the same length:

$$\mathbf{u}_1 = \frac{1}{\sqrt{20}}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{20}}\begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 = \frac{1}{\sqrt{20}}\begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

so then

$$\mathbf{Q} = \frac{1}{\sqrt{20}}\begin{bmatrix} 3 & 1 & -3 \\ 1 & 3 & 1 \\ -1 & 3 & 1 \\ 3 & -1 & 3 \end{bmatrix}, \quad \mathbf{R} = \mathbf{Q}^\top\mathbf{A} = \frac{10}{\sqrt{20}}\begin{bmatrix} 2 & -4 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

(after the workshop, check back that $\mathbf{QR} = \mathbf{A}$).

Question 3

The right-hand side of the system is given by $\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 1 \end{bmatrix}$. Then we solve $\mathbf{R}\check{\mathbf{x}} = \mathbf{Q}^\top\mathbf{b}$:

$$\mathbf{Q}^\top\mathbf{b} = \frac{1}{\sqrt{20}}\begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix} \qquad \text{so} \qquad \frac{10}{\sqrt{20}}\begin{bmatrix} 2 & -4 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}\check{\mathbf{x}} = \frac{1}{\sqrt{20}}\begin{bmatrix} 1 \\ 9 \\ 3 \end{bmatrix} \qquad \Rightarrow$$

$$\left[\begin{array}{rrr|r} 20 & -40 & 30 & 1 \\ 0 & 20 & -10 & 9 \\ 0 & 0 & 20 & 3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 7/8 \\ 0 & 1 & 0 & 21/40 \\ 0 & 0 & 1 & 3/20 \end{array}\right] \quad \Rightarrow \quad \check{\mathbf{x}} = \begin{bmatrix} 7/8 \\ 21/40 \\ 3/20 \end{bmatrix}$$

The error of this approximation is

$$\mathbf{b} - \mathbf{A}\check{\mathbf{x}} = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 1 \end{bmatrix} - \frac{1}{20}\begin{bmatrix} 3 \\ 31 \\ 29 \\ 3 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 17 \\ -51 \\ 51 \\ 17 \end{bmatrix} \quad \text{so} \quad \|\mathbf{b} - \mathbf{A}\check{\mathbf{x}}\| = \frac{17}{\sqrt{20}}$$

Question 4

Denote the matrix of this system as $\mathbf{A}$ and the right-hand side as $\mathbf{b}$.

(a) The normal equations for this system are obtained with

$$\mathbf{A}^\mathsf{T}\mathbf{A} = \begin{bmatrix} 6 & 2 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^\mathsf{T}\mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

Row-reducing the augmented matrix yields

$$\cdots \to \begin{bmatrix} 6 & 2 & -4 \\ 0 & 7/3 & 7/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \Rightarrow \quad \check{\mathbf{x}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which is the least-squares solution.

(b) The least-squares error is

$$\|\mathbf{b} - \mathbf{A}\check{\mathbf{x}}\| = \left\| \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\| = \sqrt{14}$$

(c) The columns of $\mathbf{A}$ are linearly independent, however not orthogonal; using the Gram-Schmidt process, the second vector for an orthogonal basis of $\operatorname{Col}\mathbf{A}$ is

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{-1+2+1}{1+4+1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -4/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

and then after normalising these vectors we obtain

$$\mathbf{Q} = \begin{bmatrix} \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{21}} \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \mathbf{Q}^\mathsf{T}\mathbf{A} = \begin{bmatrix} \sqrt{6} & \sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{7}{3}} \end{bmatrix}$$

(d) The alternative path to the least-squares solution is by solving $\mathbf{R}\check{\mathbf{x}} = \mathbf{Q}^\mathsf{T}\mathbf{b}$:

$$\left[\begin{array}{cc|c} \sqrt{6} & \sqrt{\frac{2}{3}} & -2\sqrt{\frac{2}{3}} \\ 0 & \sqrt{\frac{7}{3}} & \sqrt{\frac{7}{3}} \end{array}\right] \to \left[\begin{array}{cc|r} 3 & 1 & -2 \\ 0 & 1 & 1 \end{array}\right] \quad \Rightarrow \quad \check{\mathbf{x}} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

which is, of course, the same unique solution as obtained in (a).