University of Technology Sydney

School of Mathematical and Physical Sciences

37233 Linear Algebra

Final test help sheet

$(\mathbf{u}+\mathbf{v}) \in V$	$c\,\mathbf{u} \in V$
(i) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$	(v) $1 \cdot \mathbf{u} = \mathbf{u}$
(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$	(vi) $c\,(d\,\mathbf{u}) = (c\,d)\,\mathbf{u}$
(iii) $\exists\,\mathbf{0} \;:\; \mathbf{u}+\mathbf{0} = \mathbf{u}$	(vii) $(c+d)\,\mathbf{u} = c\,\mathbf{u} + d\,\mathbf{u}$
(iv) $\forall\,\mathbf{u}\;\exists\,(-\mathbf{u}) \;:\; \mathbf{u}+(-\mathbf{u}) = \mathbf{0}$	(viii) $c\,(\mathbf{u}+\mathbf{v}) = c\,\mathbf{u} + c\,\mathbf{v}$

$[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}$ using $\left[\begin{array}{cccc|cccc} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n & \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{array}\right] \to \left[\begin{array}{c|c} \mathbf{I} & \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} \end{array}\right]$

The invertible matrix theorem — equivalent statements for $n \times n$ matrix $\mathbf{A}$:

- There is an $n \times n$ matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$
- $\det \mathbf{A} \neq 0$
- $\mathbf{A}$ has n pivot positions in the REF form
- $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution
- The columns (rows) of $\mathbf{A}$ form a linearly independent set
- The columns (rows) of $\mathbf{A}$ span $\mathbb{R}^n$
- The columns (rows) of $\mathbf{A}$ form a basis of $\mathbb{R}^n$
- $\widehat{T}: \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is one-to-one
- $\widehat{T}: \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
- The range of $\widehat{T}: \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is $\mathbb{R}^n$
- $\operatorname{Col}\mathbf{A} = \operatorname{Row}\mathbf{A} = \mathbb{R}^n$
- $\operatorname{Nul}\mathbf{A} = \{\mathbf{0}\}$ and $\dim(\operatorname{Nul}\mathbf{A}) = 0$
- $\dim(\operatorname{Col}\mathbf{A}) = \dim(\operatorname{Row}\mathbf{A}) = n$
- $\operatorname{rank}\mathbf{A} = n$
- The eigenvalues of $\mathbf{A}$ are non-zero
- $\mathbf{A}^{\mathsf{T}}$ is invertible

$$\begin{aligned} l_{kk} &= 1 && k = 1, 2, \ldots, n \\ u_{kj} &= a_{kj} - \sum_{m=1}^{k-1} l_{km} u_{mj} && k \leqslant j \leqslant n \\ l_{ik} &= \frac{1}{u_{kk}} \left(a_{ik} - \sum_{m=1}^{k-1} l_{im} u_{mk} \right) && k \leqslant i \leqslant n \end{aligned}$$

$$\begin{aligned} u_{ii} &= 1 && i = 1, 2, \ldots, n \\ u_{ij} &= \frac{1}{l_{ii}} \left(a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \right) && i < j = 2, 3, \ldots, n. \\ l_{ij} &= a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} && i \geqslant j = 1, 2, \ldots, n \end{aligned}$$

$$\begin{aligned} \text{For} \quad i > j : \qquad & u_{ij} = 0 \\ \text{For} \quad j = 2,\, 3, \ldots, n : \qquad & u_{1j} = \frac{a_{1j}}{\sqrt{a_{11}}} \\ \text{For} \quad i = 1,\, 3, \ldots, n : \qquad & u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2} \\ \begin{cases} \text{For} \quad i = 2,\, 3, \ldots, n \\ \qquad\;\; j = i+1,\, i+2, \ldots, n \end{cases} : \qquad & u_{ij} = \frac{1}{u_{ii}} \left(a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj} \right) \end{aligned}$$

$$\mathbf{v}_i = \mathbf{x}_i - \sum_{j=1}^{i-1} \left(\frac{\mathbf{x}_i \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j} \, \mathbf{v}_j \right)$$

$$\mathbf{A} = \mathbf{Q}\mathbf{R} \qquad : \qquad \mathbf{R}\check{\mathbf{x}} = \mathbf{Q}^{\mathsf{T}}\mathbf{b} \qquad \Longleftrightarrow \qquad \mathbf{A}^{\mathsf{T}}\mathbf{A}\check{\mathbf{x}} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$$

$$m \times n \quad \text{matrix} \qquad \mathbf{A} = \mathbf{W}\Sigma\mathbf{U}^{\mathsf{T}} = \begin{bmatrix} \mathbf{w}_1 \ldots \mathbf{w}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & 0 & 0 & \\ & \ddots & & 0 & \\ 0 & & \sigma_r & 0 & \ldots \\ 0 & 0 & 0 & 0 & \\ & & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^{\mathsf{T}} \\ \vdots \\ \mathbf{u}_n^{\mathsf{T}} \end{bmatrix} = \sum_{i=1}^{r} \sigma_i \mathbf{w}_i \mathbf{u}_i^{\mathsf{T}}$$

where $r = \operatorname{rank} \mathbf{A}$, and $\sigma_i = \sqrt{\lambda_i}$ $\big(\sigma_i \geqslant \sigma_{i+1} > 0\big)$, and $\mathbf{u}_i$ are normalised eigenvectors of $\mathbf{A}^{\mathsf{T}}\mathbf{A}$, and $\mathbf{w}_i = \dfrac{\mathbf{A}\mathbf{u}_i}{\sigma_i}$ for $1 \leqslant i \leqslant r$, extended to an orthonormal basis of $\mathbb{R}^m$ for $r < i \leqslant m$.