37233 Linear Algebra

Revision of the essential pre-requisite topics

Revision: vectors, matrices, linear systems

- Basics of vector algebra
- Basics of matrix algebra
- Matrix inverse and determinants
- Systems of linear equations
- Gaussian reduction
- Homogeneous and inhomogeneous linear systems

Numbers can be organised as scalars, vectors, matrices, and further on as higher-order *tensors*.

- Scalar a is a single number.
- Vector $\mathbf{a}$ ($\equiv \vec{a}$) is an ordered list of numbers. For example

$$\mathbf{a} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

  Vector with all entries equal to 0 is called a *zero vector* : $\mathbf{0}$

- Matrix is a two-dimensional array of numbers. For example

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$$

(only defined for vectors with the same number n of components)

Equality: if all the corresponding components are equal

$$\mathbf{v} = \mathbf{u} \qquad \text{if} \qquad v_i = u_i \qquad \forall i = 1 \dots n$$

Multiplication by a scalar: each component is multiplied

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad c\,\mathbf{v} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}$$

Addition: corresponding components are added

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} + \mathbf{u} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix}$$

Matrix operations

- Addition and multiplication by a scalar

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}+b_{11}) & (a_{12}+b_{12}) \\ (a_{21}+b_{21}) & (a_{22}+b_{22}) \end{bmatrix}$$

$$c \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} c\,a_{11} & c\,a_{12} \\ c\,a_{21} & c\,a_{22} \end{bmatrix}$$

- Multiplying matrices by vectors

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 \\ a_{21}b_1 + a_{22}b_2 \\ a_{31}b_1 + a_{32}b_2 \end{bmatrix}$$

(vector size must be equal to the number of matrix <u>columns</u>)

Matrix multiplication

Multiplying $l \times m$ matrix $\mathbf{A}$ by $m \times n$ matrix $\mathbf{B}$ results in $l \times n$ matrix

with elements $$\{\mathbf{AB}\}_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj} = \sum_{k=1}^{m} a_{ik}b_{kj}$$

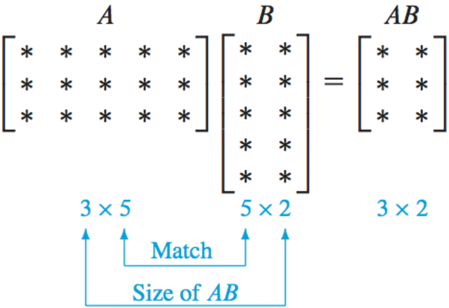
Examples: $$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{21} + a_{12}b_{22}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{21} + a_{22}b_{22}) \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}) \end{bmatrix}$$

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} (b_{11}a_{11} + b_{12}a_{21}) & (b_{11}a_{12} + b_{12}a_{22}) & (b_{11}a_{13} + b_{12}a_{23}) \\ (b_{21}a_{11} + b_{22}a_{21}) & (b_{21}a_{12} + b_{22}a_{22}) & (b_{21}a_{13} + b_{22}a_{23}) \\ (b_{31}a_{11} + b_{32}a_{21}) & (b_{31}a_{12} + b_{32}a_{22}) & (b_{31}a_{13} + b_{32}a_{23}) \end{bmatrix}$$

Note that the order of multiplication is essential: $\mathbf{AB} \neq \mathbf{BA}$

Matrix multiplication



Example:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix} =$$

$$\begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} + a_{14}b_{41}) & (a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + a_{14}b_{42}) \\ (a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} + a_{24}b_{41}) & (a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + a_{24}b_{42}) \\ (a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} + a_{34}b_{41}) & (a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} + a_{34}b_{42}) \end{bmatrix}$$

(note that in these examples $\mathbf{BA}$ product is not even possible)

Multiplication and transposition

For matrix multiplication (provided that the sizes are appropriate):

- $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
- $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
- $c \cdot (\mathbf{AB}) = (c \cdot \mathbf{A})\mathbf{B} = \mathbf{A}(c \cdot \mathbf{B})$

Transposition $\mathbf{A}^\mathsf{T}$ of a matrix $\mathbf{A}$ exchanges its rows and columns:

$$\{\mathbf{A}^\mathsf{T}\}_{ij} = \{\mathbf{A}\}_{ji}$$

Example: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $\mathbf{A}^\mathsf{T} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

For any (multipliable) matrices: $(\mathbf{AB})^\mathsf{T} = \mathbf{B}^\mathsf{T}\mathbf{A}^\mathsf{T}$

Unitary (identity) matrix $\mathbf{I}$ is a square matrix with all the main diagonal entries equal to 1, and all the other entries equal to 0

Examples of different size:

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{I}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplication of a matrix with $\mathbf{I}$ results in the same matrix:

$$\mathbf{AI} = \mathbf{A} \qquad \text{as well as} \qquad \mathbf{IA} = \mathbf{A}$$

(so it plays the role of unity for matrix multiplications)

Definition: An $n \times n$ matrix $\mathbf{A}$ is called invertible if there is an $n \times n$ matrix, denoted $\mathbf{A}^{-1}$, such that

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \quad \text{and} \quad \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

(where $\mathbf{I}$ is a unitary $n \times n$ matrix).

A non-invertible matrix is called a **singular** matrix.

Theorem:

a) If $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1}$ is invertible, and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$.

b) If $\mathbf{A}$ and $\mathbf{B}$ are invertible, then $(\mathbf{A}\mathbf{B})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

c) If $\mathbf{A}$ is invertible, then $\left(\mathbf{A}^{\mathsf{T}}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\mathsf{T}}$

Easy for a 2×2 matrix: $\det \mathbf{A}_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

For a 3×3 matrix, this is somewhat more involved:

$$\det \mathbf{A}_3 = a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

and clearly gets overwhelmingly elaborate as the matrix size grows:

For a square $n \times n$ matrix $\mathbf{A}$ with elements a_{ij}, the determinant is

$$\det \mathbf{A} = \sum^{n!} \left((-1)^{N(\alpha_1, \alpha_2 \ldots \alpha_n)} \prod_{k=1}^{n} a_{k\alpha_k} \right) \quad \Bigg| \quad \begin{matrix} \alpha_i \in [1,\, n] \\ \alpha_i \neq \alpha_j \end{matrix}$$

where the sum is over all the $n!$ permutations of numbers 1 to n, and $N(\alpha_1, \alpha_2 \ldots \alpha_n)$ is the amount of *disorders* in each $\alpha_1 \ldots \alpha_n$ index sequence: the amount of cases $\{\, \alpha_i > \alpha_j$ for $i < j \,\}$.

Useful determinant properties

- $\det \mathbf{A}^\mathsf{T} = \det \mathbf{A}$
- $\det(\mathbf{A}\mathbf{B}) = \det \mathbf{A} \cdot \det \mathbf{B}$
- $\det \mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}}$ (if $\mathbf{A}$ is invertible)
- $\det \mathbf{I} = 1$ (but $\det \mathbf{A} = 1$ does not mean $\mathbf{A} = \mathbf{I}$)
- $\mathbf{A}$ is invertible if and only if $\det \mathbf{A} \neq 0$

  (so if determinant $\det \mathbf{A} = 0$, then $\mathbf{A}$ is singular)

Useful determinant properties

- If two rows are exchanged, determinant changes sign

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- If two rows are identical, determinant equals to zero

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$$

- Common scalar multiplier of a row can be taken out as multiplier

$$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

- If all elements of a row are zero, determinant is zero

$$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$$

- If a row multiplied by a scalar is added to another row, determinant does not change

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c + ka & d + kb \end{vmatrix}$$

(same rules are applicable with respect to columns)

Revision: Linear equations

- Linear equation in variables $x_1, x_2, \ldots, x_n$ is an equation

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

where b and $a_1, a_2, \ldots a_n$ are real (or complex) numbers.

- For example, equations

$$4x_1 - 5x_2 + 2 = x_1, \qquad x_2 = 2(\sqrt{6} - x_1) + x_3$$

are linear and can be arranged in the above form:

$$3x_1 - 5x_2 = -2, \qquad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

- The following examples

$$4x_1 - 5x_2 + 2 = x_1 \sin x_2, \qquad x_2 = 2\sqrt{x_1} - 6$$

are not linear equations.

Systems of linear equations

- A system of linear equations (a linear system) is a collection of linear equations involving the same variables:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & \dots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m. \end{cases}$$

- The solution of the system is a list of variables $x_1, x_2, \dots, x_n$ that makes *each* equation a true statement.

- A linear system may have

  - infinitely many solutions

  - exactly one solution

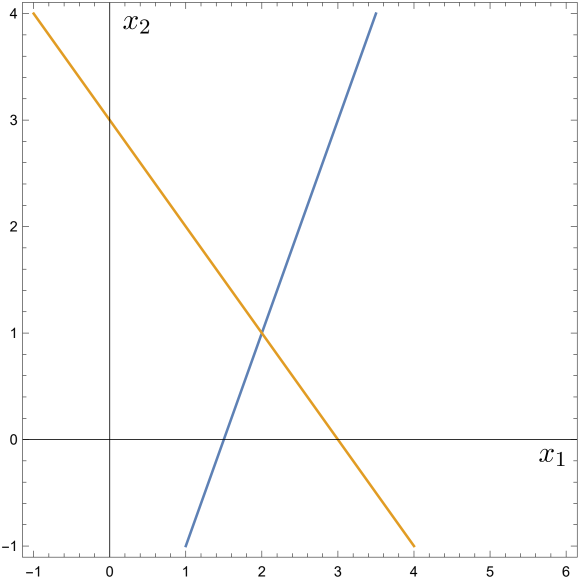
  - no solutions

Consider an example

$$\begin{cases} 2x_1 & - & x_2 & = & 3 \\ x_1 & + & x_2 & = & 3 \end{cases}$$

By plotting the above equations in $\{x_1, x_2\}$ plane, we find the unique solution of this system as:
$x_1 = 2$ and $x_2 = 1$.

That's the "old way"



However, we can rewrite this linear system as follows:

$$\begin{cases} 2x_1 - x_2 = 3 \\ x_1 + x_2 = 3 \end{cases} \Leftrightarrow \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

which is equivalent to

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

So, the original system is rewritten as a linear combination

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$$

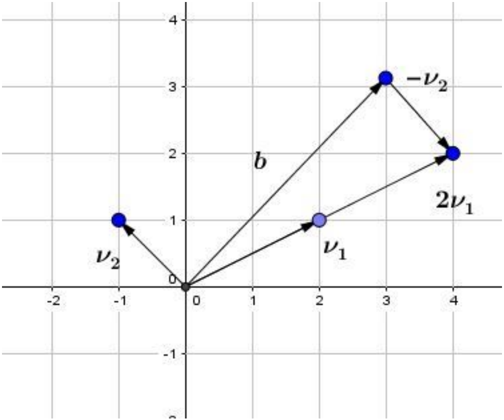
where

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{cases} 2x_1 - x_2 = 3 \\ x_1 + x_2 = 3 \end{cases}$$

Graphical visualisation for the unique solution $x_1 = 2$ and $x_2 = 1$:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{b}$$

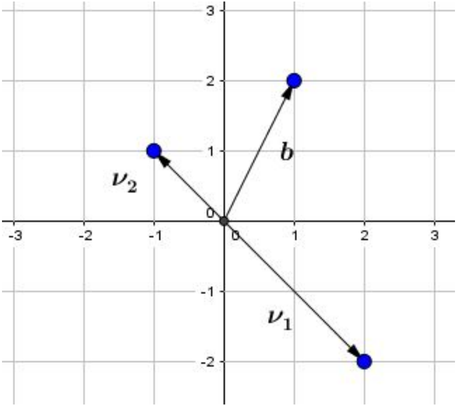


$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Another example:

$$\begin{cases} 2x_1 & - & x_2 & = & 1 \\ -2x_1 & + & x_2 & = & 2 \end{cases}$$

Here, no solutions.



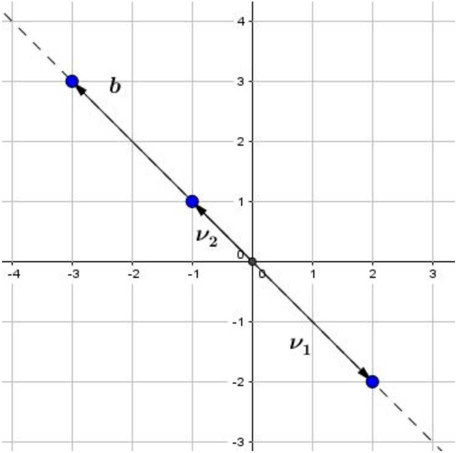
$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = x_1\begin{bmatrix} 2 \\ -2 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{b}$$

Another example:

$$\begin{cases} 2x_1 & - & x_2 & = & -3 \\ -2x_1 & + & x_2 & = & 3 \end{cases}$$

Infinite number of solutions

$(x_2 = 2x_1 + 3 \quad \forall x_1)$



$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = x_1 \begin{bmatrix} 2 \\ -2 \end{bmatrix} + (2x_1 + 3) \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = \mathbf{b}$$

A linear system with m equations and n unknowns

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1, \\ a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & = & b_2, \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & = & b_m. \end{cases}$$

can be written using $m \times n$ matrix $\mathbf{A}$ of the coefficients a_{ij}, vector $\mathbf{x}$ of variables x_i, and vector $\mathbf{b}$ of the right-hand side:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Then the entire system reads: $\mathbf{A}\mathbf{x} = \mathbf{b}$

Theorem:

If $\mathbf{A}$ is an invertible $n \times n$ matrix then $\forall\, \mathbf{b}$ with n components, equation $\mathbf{Ax} = \mathbf{b}$ has unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

However, using this theorem for solving the system is limited to square matrices, and is rather inefficient.

Instead, we form an augmented matrix of the system

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right]$$

and apply Gaussian reduction.

Gaussian reduction / elimination

In order to solve a linear system using matrix representation, we need to reduce the augmented matrix to an *echelon form*.

This process is called Gauss – Jordan elimination, achieved with a series of *row operations*.

Row operations that can be used:

- Swapping two rows
- Multiplying a row by a constant
- Adding a multiple of one row to another

(schematic example: $\blacksquare$ is a non-zero number, and $*$ is any number)

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first non-zero element in a row is called the **leading element**

The matrix in **echelon form** has the following properties:

- All non-zero rows are above any rows of all zeros.
- Each leading entry of a row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeros.

Reduced echelon form (REF)

The next step is to obtain a reduced echelon form (REF):

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

REF matrix, in addition to **EF** form, has the properties that

- the leading entry in each non-zero row is 1
- the leading 1 is the only non-zero entry in its column

The echelon form of a matrix (**EF**) is not unique,
however reduced echelon form (**REF**) is unique:

Each matrix is row-equivalent to one and only
one matrix in reduced echelon form (REF).

Schematic examples of an EF form

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Schematic examples of a REF form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

A **pivot position** corresponds to the leading (non-zero) entry.

A **pivot column** is a column that contains a pivot position.

Finding solutions of a linear system using Gaussian reduction:

$$\left\{\begin{array}{rcrcrcrcrcr} & & 3x_2 & - & 6x_3 & + & 6x_4 & + & 4x_5 & = & -5 \\ 3x_1 & - & 7x_2 & + & 8x_3 & - & 5x_4 & + & 8x_5 & = & 9 \\ 3x_1 & - & 9x_2 & + & 12x_3 & - & 9x_4 & + & 6x_5 & = & 15 \end{array}\right.$$

Write the augmented matrix of this system:

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$$

Follow the protocol, using elementary row operations

- Swapping two rows
- Multiplying a row by a constant
- Adding a multiple of one row to another

Row reduction to EF (example)

Starting matrix: $\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right]$

- **Step 1**: Begin with the first left non-zero column.
  Swap the rows to bring any zeros of the first column down.
  Select nonzero entry in the pivot column as a pivot.

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

- **Step 2**: Use row replacement operation to create zeros in all positions below the pivot (here, use $R_2 \to R_2 - R_1$).

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

Row reduction to EF (example)

- **Step 3**. Cover the row containing the pivot (and any rows above it). Apply steps 1–2 to the remaining sub-matrix. Repeat until there are no more non-zero rows to modify.

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

(now, we divide the second row by 2 for convenience)

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array}\right]$$

(now, we use $\mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_2$)

$$\left[\begin{array}{ccccc|c} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Row reduction from EF to REF (example)

We have achieved an EF:

$$\left[\begin{array}{ccccc|c} \boxed{3} & -9 & 12 & -9 & 6 & 15 \\ 0 & \boxed{1} & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & \boxed{1} & 4 \end{array}\right]$$

This is the end of the "forward" phase (down and to the right).

From here, we will work "backward" (to the left and up).

- **Step 4** : If a pivot is not 1, make it 1 by a scaling operation. Here, row operation $R_1 \to R_1/3$ leads to

$$\left[\begin{array}{ccccc|c} \color{red}{1} & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

- **Step 5** : Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot.

  Here, $\mathsf{R}_2 \to \mathsf{R}_2 - \mathsf{R}_3$ leads to

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

  and $\mathsf{R}_1 \to \mathsf{R}_1 - 2\mathsf{R}_3$ leads to

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

The REF form ("backward phase")

- Done with the lowest-right pivot; address the next left-up

$$\left[\begin{array}{ccccc|c} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

by making $\mathsf{R}_1 \to \mathsf{R}_1 + 3\mathsf{R}_2$:

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

This finally brings the matrix to the REF form.

The resulting matrix corresponds to an equivalent system

$$\begin{aligned} x_1 - 2x_3 + 3x_4 &= -24 \\ x_2 - 2x_3 + 2x_4 &= -7 \\ x_5 &= 4 \end{aligned}$$

REF form and system solution

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \qquad \begin{array}{rcl} x_1 - 2x_3 + 3x_4 & = & -24 \\ x_2 - 2x_3 + 2x_4 & = & -7 \\ x_5 & = & 4 \end{array}$$

- Note there are 3 equations for 5 variables.
- Variables with pivots are called **basic variables**: x_1, x_2, x_5.
- Variables without pivots are called **free variables**: x_3, x_4.
- In the final solution basic variables (here, x_1, x_2, x_5) must be expressed via free variables (here, x_3, x_4).

The solution is:

$$\begin{array}{rcl} x_1 & = & -24 + 2x_3 - 3x_4 \\ x_2 & = & -7 + 2x_3 - 2x_4 \\ x_5 & = & 4 \end{array}$$

The solution can be written in the form of variables:

$$\left[\begin{array}{cccccc} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \quad \Leftrightarrow \quad \begin{array}{lcl} x_1 & = & -24 + 2x_3 - 3x_4 \\ x_2 & = & -7 + 2x_3 - 2x_4 \\ x_5 & = & 4 \end{array}$$

...or in the form of a solution vector:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -24 + 2x_3 - 3x_4 \\ -7 + 2x_3 - 2x_4 \\ x_3 \\ x_4 \\ 4 \end{bmatrix} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Example with a unique solution

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Upon a sequence of row operations ...

$$\begin{array}{cl} \text{Eq3} + 4 * \text{Eq1} \to \text{Eq3} & \text{R3} + 4 * \text{R1} \to \text{R3} \\ \text{Eq2} \to 0.5 \times \text{Eq2} & \text{R2} \to 0.5 \times \text{R2} \\ \text{Eq3} + 3 * \text{Eq2} \to \text{Eq3} & \text{R3} + 3 * \text{R2} \to \text{R3} \\ \text{Eq1} + 2 * \text{Eq2} \to \text{Eq1} & \text{R1} + 2 * \text{R2} \to \text{R1} \\ \text{Eq1} + 7 * \text{Eq3} \to \text{Eq1} & \text{R1} + 7 * \text{R2} \to \text{R1} \\ \text{Eq2} + 4 * \text{Eq3} \to \text{Eq2} & \text{R2} + 4 * \text{R3} \to \text{R2} \end{array}$$

... we arrive at a unique solution:

$$\begin{array}{ccccc} x_1 & & & = & 29 \\ & x_2 & & = & 16 \\ & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example with no solutions

$$\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array} \qquad \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix}$$

Upon doing $\mathsf{R}_1 \leftrightarrow \mathsf{R}_2$ and $\mathsf{R}_3 \to (2\mathsf{R}_3 - 5\mathsf{R}_2) + \mathsf{R}_1$ we get

$$\begin{array}{rcrcrcl} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 8 \\ & & & & 0 & = & 5 \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

The **inconsistency** $0 \stackrel{!}{=} 5$ implies that this system does not have any solutions.

Example with infinitely many solutions

$$\begin{array}{rcrcrcr} & & x_2 & - & 4x_3 & = & 6 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & -1/2 \end{array} \qquad \begin{bmatrix} 0 & 1 & -4 & 6 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & -1/2 \end{bmatrix}$$

Doing again $\mathsf{R}_1 \leftrightarrow \mathsf{R}_2$ and $\mathsf{R}_3 \to (2\mathsf{R}_3 - 5\mathsf{R}_2) + \mathsf{R}_1$ yields

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 6 \\ & & & & 0 & = & 0 \end{array} \qquad \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3 equations in 3 unknowns $\longrightarrow$ 2 equations in 3 unknowns
$\Rightarrow$ only two independent equations

No contradiction, but no unique solution (infinitely many)

- Infinitely many, or a unique solution — system is **consistent**
- No solutions — system is **inconsistent**

(example 1): Consistent system, unique solution, e.g.:

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(example 2): Inconsistent system, no solution, e.g.:

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

(example 3): Consistent system, infinitely many solutions, e.g.:

$$\begin{bmatrix} 0 & 1 & -4 & 6 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & -1/2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathbf{A}^{-1}$ by Gaussian elimination

An $n \times n$ matrix $\mathbf{A}$ is invertible if and only if it is row-equivalent to identity matrix $\mathbf{I}$, and a sequence of elementary row operations that reduces $\mathbf{A}$ to $\mathbf{I}$, transforms $\mathbf{I}$ into $\mathbf{A}^{-1}$.

This gives an easy algorithm for calculating $\mathbf{A}^{-1}$:

Row reduce the augmented matrix $[\mathbf{A} \,|\, \mathbf{I}\,]$.

If $\mathbf{A}$ is row-reduced to $\mathbf{I}$, then $[\mathbf{A} \,|\, \mathbf{I}\,]$ is row-reduced to $[\mathbf{I} \,|\, \mathbf{A}^{-1}]$.

Otherwise $\mathbf{A}$ does not have an inverse.

Example: find inverse of the matrix below, if it exists

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$$

$\mathbf{A}^{-1}$ by Gaussian elimination

We form an augmented $[\mathbf{A} \mid \mathbf{I}]$ matrix, and try row-reducing:

$$\left[\begin{array}{cccccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\Downarrow$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$\mathsf{R}_1 \leftrightarrow \mathsf{R}_2$$
$$\mathsf{R}_3 \to \mathsf{R}_3 - 4\mathsf{R}_1$$
$$\mathsf{R}_3 \to \mathsf{R}_3 + 3\mathsf{R}_2$$
$$\mathsf{R}_3 \to \mathsf{R}_3/2$$
$$\mathsf{R}_2 \to \mathsf{R}_2 - 2\mathsf{R}_3$$
$$\mathsf{R}_1 \to \mathsf{R}_1 - 3\mathsf{R}_3$$

Thus $\mathbf{A}$ is invertible and $\mathbf{A}^{-1} = \left[\begin{array}{ccc} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{array}\right]$

Square matrices with all elements below (or above) the main diagonal being zero, are called triangular:

$$\mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

For a triangular matrix: $\det \mathbf{U} = \prod_{i=1}^{n} u_{ii}$

Recall the general properties of determinants:

- If two rows are exchanged, the determinant changes sign
- If a row multiplied by a scalar is added to another row, $a_{ij} \rightarrowtail a_{ij} + c \cdot a_{kj}$, the determinant does not change

We can perform row operations (without row scaling) to achieve an echelon form, which results in a triangular matrix: $\mathbf{A} \rightarrowtail \mathbf{U}$.

Then: $\det \mathbf{A} = (-1)^N \cdot \det \mathbf{U}$

where N is the number of row swaps required to obtain $\mathbf{U}$.

Example: calculate the determinant of $\mathbf{A}$ using row reduction:

$$\mathbf{A} = \begin{bmatrix} 2 & -4 & -8 & 6 \\ -3 & 6 & 12 & -8 \\ 5 & -10 & -15 & 6 \\ -1 & 12 & 7 & 4 \end{bmatrix}$$

$$\Downarrow$$

$$\mathbf{U} = \begin{bmatrix} 2 & -4 & -8 & 6 \\ 0 & 10 & 3 & 7 \\ 0 & 0 & 5 & -9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathsf{R}_2 \to \mathsf{R}_2 + 1.5 \cdot \mathsf{R}_1$$
$$\mathsf{R}_3 \to \mathsf{R}_3 - 2.5 \cdot \mathsf{R}_1$$
$$\mathsf{R}_4 \to \mathsf{R}_4 + 0.5 \cdot \mathsf{R}_1$$
$$\mathsf{R}_2 \leftrightharpoons \mathsf{R}_4$$

One row swap and no row multiplications, so

$$\det \mathbf{A} = (-1)^1 \cdot \det \mathbf{U} = -1 \cdot (2 \cdot 10 \cdot 5 \cdot 1) = -100$$

Homogeneous and inhomogeneous linear systems

A system of linear equations is *homogeneous* if it has the form

$$\mathbf{Ax} = \mathbf{0}$$

A system of linear equations is *inhomogeneous* if it has the form

$$\mathbf{Ax} = \mathbf{b} \qquad \text{with} \qquad \mathbf{b} \neq \mathbf{0}$$

A homogeneous system always has at least one solution: $\mathbf{x} = \mathbf{0}$
This solution is called a *trivial solution*.

A homogeneous system has a non-trivial solution $\mathbf{x} \neq \mathbf{0}$
if and only if there is at least one free variable.

Example:

$$\begin{array}{rcrcrcl} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So x_1 and x_2 are basic variables, and x_3 is a free variable:

$$x_1 = (4/3) \cdot x_3 \qquad \text{and} \qquad x_2 = 0 = 0 \cdot x_3$$

The solution set is:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (4/3)x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

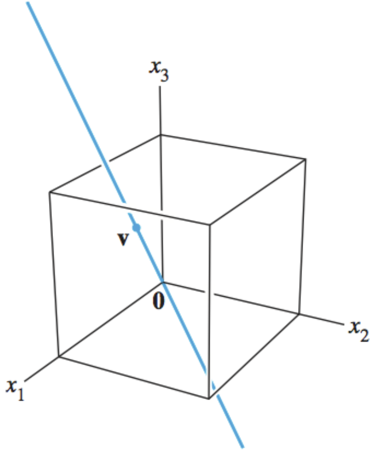
So every solution is scalar multiple

$$\mathbf{x} = t\,\mathbf{v} \qquad \forall\, t \in \mathbb{R}$$

of (any multiple of) vector

$$\mathbf{v} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

Geometrically, this solution set represents a line which passes though $\mathbf{0}$ and $\mathbf{v}$.



Example: Describe all the solutions of the homogeneous equation:

$$10x_1 - 3x_2 - 2x_3 = 0$$

A general solution is $x_1 = 0.3x_2 + 0.2x_3$. In a vector form, that is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.3x_2 + 0.2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}$$

which is

$$\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$$

where

$$\mathbf{u} = \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}.$$

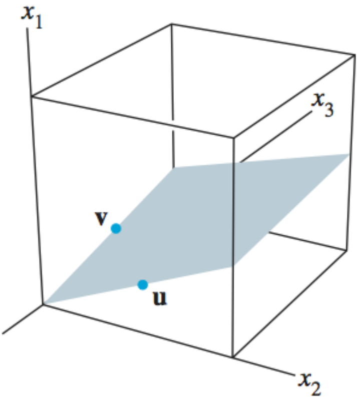
Note: one can also express x_2 or x_3 through the other variables.

This solution set

$$\mathbf{x} = x_2\mathbf{u} + x_3\mathbf{v}$$

is a *parametric equation* of a plane through the origin, defined by

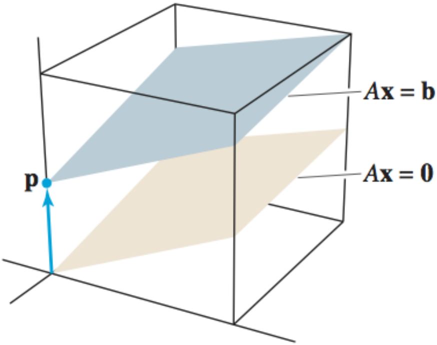
$$\mathbf{u} = \begin{bmatrix} 0.3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix}$$



Theorem:

Suppose $\mathbf{Ax} = \mathbf{b}$ is consistent for some $\mathbf{b}$ and let $\mathbf{p}$ be a solution (called a *particular* solution).

Then the solution set of $\mathbf{Ax} = \mathbf{b}$ is a set of all vectors in the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_0$, where $\mathbf{v}_0$ is any solution of the homogeneous equation $\mathbf{Ax} = \mathbf{0}$.



(illustration for three equations)

Proof: $\mathbf{Aw} = \mathbf{A}(\mathbf{p} + \mathbf{v}_0) = \mathbf{Ap} + \mathbf{Av}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}$

Example: the earlier system with a non-zero right-hand side

$$\left[\begin{array}{ccc|c} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So x_3 is free (as it was), $x_2 = 2$, and $x_1 = (4/3)x_3 - 1$.

In vector form, the solution can be written as

$$\mathbf{x} = \begin{bmatrix} \frac{4}{3}x_3 - 1 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

which is $\quad \mathbf{x} = \mathbf{p} + x_3\mathbf{v} \quad$ with $\quad \mathbf{p} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}.$

Recall the solution of the homogeneous system was $\mathbf{x} = x_3\mathbf{v}$.

Vector $\mathbf{p}$ is a particular solution of $\mathbf{A}\mathbf{x} = \mathbf{b}$ (for $x_3 = 0$).

Another example:

$$\left[\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

Basic variables are x_1, x_2, x_5 and free variables are x_3, x_4.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 & - & 3x_4 & - & 24 \\ 2x_3 & - & 2x_4 & - & 7 \\ x_3 & & & & \\ & & x_4 & & \\ & & & & 4 \end{bmatrix} = x_3\mathbf{u} + x_4\mathbf{v} + \mathbf{p}$$

where $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{p} = \begin{bmatrix} -24 \\ -7 \\ 0 \\ 0 \\ 4 \end{bmatrix}.$

Thereby, $\mathbf{u}$ and $\mathbf{v}$ are solutions to the homogeneous system

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= 0 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 9x_5 &= 0 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 0 \end{aligned}$$

Its general solution is: $\mathbf{x} = x_3\mathbf{u} + x_4\mathbf{v}$.

Then, $\mathbf{p}$ is a particular solution to the inhomogeneous system

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 9x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

which is obtained by specifying (for simplicity) $x_3 = x_4 = 0$.

The general solution here is: $\mathbf{x} = \mathbf{p} + x_3\mathbf{u} + x_4\mathbf{v}$.

General solutions for homogeneous and inhomogeneous systems:

- Row-reduce the augmented matrix to REF
- Express each basic variable in terms of free variables
- Write the solution in vector form, using the free variables as coefficients (weights)
- General solution of the homogeneous system is described by the vectors which stand at these coefficients
- A particular solution of the inhomogeneous system is a fixed vector (most easily obtained by setting free variables to zero)
- General inhomogeneous system solution is the sum of the particular solution and the homogeneous general solution

Summary

We have now refreshed your knowledge on:

- Vectors and matrices
- Systems of linear equations
- Gaussian reduction
- Homogeneous and inhomogeneous systems

Ability to solve linear systems is crucial for the rest of this subject.

We will then continue with fundamentals of linear algebra.