University of Technology Sydney
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37233 Linear Algebra

Tutorial 2 — solutions guide

Question 1

Operations $\mathbf{a}+\mathbf{b}$, $\mathbf{Aa}$, $\mathbf{Bb}$, $\mathbf{A}+\mathbf{B}$ and $\mathbf{AB}^\mathsf{T}$ are not defined, and the result of the rest is

$$\mathbf{Ab} = \begin{bmatrix} -5 \\ 2 \\ 11 \end{bmatrix}; \quad \mathbf{Ba} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \mathbf{A}+\mathbf{B}^\mathsf{T} = \begin{bmatrix} 1 & 4 \\ 5 & 2 \\ 3 & 2 \end{bmatrix}; \quad \mathbf{AB} = \begin{bmatrix} 3 & 0 & 1 \\ 9 & 6 & -1 \\ 7 & 8 & -3 \end{bmatrix}; \quad \mathbf{BA} = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}$$

Question 2

(a) Row reduction (please do step-by-step) will eventually result in

$$\begin{bmatrix} 1 & -2 & 0 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & -5 & -3 \\ 0 & 0 & 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots are found in columns 1, 3 and 4, so x_1, x_3 and x_4 are basic variables. There are no pivots in columns 2 and 5, so x_2 and x_5 are free variables. Expressing the basic variables through the free variables, the solution is:

$$\begin{aligned} x_1 &= 2 + 2x_2 - 3x_5 \\ x_3 &= -3 + 5x_5 \\ x_4 &= 7 - x_5 \end{aligned}$$

which can be also expressed in the vector form as

$$\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 7 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 5 \\ -1 \\ 1 \end{bmatrix}$$

(b) Row reduction (please do step-by-step) will eventually result in

$$\begin{bmatrix} 1 & 0 & -1 & 3 & 0 \\ 0 & 1 & 2 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

which reveals an inconsistency ($0 \neq 1$), so the system has no solutions.

(c) Row reduction (please do step-by-step) will eventually result in

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which indicates a unique solution $x_1 = -3$, $x_2 = 1$, $x_3 = 5$.

Question 3

(a) Row reduction of the augmented matrix $\left[\,\mathbf{A}\,|\,\mathbf{I}\,\right]$ will eventually produce

$$\begin{bmatrix} 2 & 4 & 6 & 1 & 0 & 0 \\ 4 & 5 & 5 & 0 & 1 & 0 \\ 3 & 1 & -3 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & -10 & 9 & -5 \\ 0 & 1 & 0 & 13.5 & -12 & 7 \\ 0 & 0 & 1 & -5.5 & 5 & -3 \end{bmatrix}$$

so then the inverse matrix is

$$\mathbf{A}^{-1} = \begin{bmatrix} -10 & 9 & -5 \\ 13.5 & -12 & 7 \\ -5.5 & 5 & -3 \end{bmatrix}$$

(b)

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} -7 \\ 10.5 \\ -4.5 \end{bmatrix}$$

Question 4

Make row reduction to echelon form withiout row multiplications. Subtracting the first row, multiplied so as to eliminate the first column, leads to the matrix shown on the left; subsequent row swaps ($4 \leftrightarrow 3$ then $3 \leftrightarrow 2$) bring it to the matrix shown on the right:

$$\begin{bmatrix} -2 & 4 & -6 & 8 \\ 0 & 0 & 10 & -9 \\ 0 & 0 & 0 & 5 \\ 0 & 1 & 3 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 4 & -6 & 8 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 10 & -9 \\ 0 & 0 & 0 & 5 \end{bmatrix} \equiv \mathbf{U}$$

For the triangular matrix $\mathbf{U}$, the determinant is equal to $\det \mathbf{U} = -2 \cdot 1 \cdot 10 \cdot 5 = -100$. There have been 2 row swaps during reduction, so then

$$\det \mathbf{A} = (-1)^2 \cdot \det \mathbf{U} = -100$$

Question 5

First, we establish that these matrices are square matrices. Suppose $\mathbf{A}$ is $i \times j$, $\mathbf{B}$ is $k \times l$ and $\mathbf{C}$ is $m \times n$. Given that $\mathbf{BA} = \mathbf{B}$ we conclude that $l = i$ and $l = j$, so then $i = j$. Given that $\mathbf{BC} = \mathbf{A}$ we conclude that $k = i$ and $n = j$ and $m = l$, so then $n = i$ and $m = i$. Thus, all the matrices are square matrices of the same size.

Since $\mathbf{B}$ is invertible, we can multiply $\mathbf{BA} = \mathbf{B}$ by $\mathbf{B}^{-1}$ from the left:

$$\mathbf{BA} = \mathbf{B} \quad \Rightarrow \quad \mathbf{B}^{-1}\mathbf{BA} = \mathbf{B}^{-1}\mathbf{B} \quad \Rightarrow \quad \mathbf{IA} = \mathbf{I} \quad \Rightarrow \quad \mathbf{A} = \mathbf{I}$$

Then $\mathbf{C}$ is the inverse of $\mathbf{B}$ since

$$\mathbf{BC} = \mathbf{A} \quad \Rightarrow \quad \mathbf{B}^{-1}\mathbf{BC} = \mathbf{B}^{-1}\mathbf{A} \quad \Rightarrow \quad \mathbf{IC} = \mathbf{B}^{-1}\mathbf{I} \quad \Rightarrow \quad \mathbf{C} = \mathbf{B}^{-1}$$

All the remaining relationships can be then quickly recovered based on that.

For example, multiply $\mathbf{BB} = \mathbf{C}$ by $\mathbf{C}$ from the right:

$$\mathbf{BB} = \mathbf{C} \quad \Rightarrow \quad \mathbf{BBC} = \mathbf{CC} \quad \Rightarrow \quad \mathbf{BI} = \mathbf{CC} \quad \Rightarrow \quad \mathbf{CC} = \mathbf{B}$$

(a) The complete table is shown on the right

(b) The identity matrix is $\mathbf{A}$

(c) The inverse of $\mathbf{C}$ is $\mathbf{B}$

(d) $\mathbf{BAC} = \mathbf{BC} = \mathbf{A}$

Product	Result
AA	**A**
AB	**B**
AC	**C**
BA	**B**
BB	**C**
BC	**A**
CA	**C**
CB	**A**
CC	**B**

Question 6

Suppose $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and express the matrix product $\mathbf{A}^2 = \mathbf{AA}$ explicitly.

Equating the result, term by term, to $-\mathbf{I}$, and analysing the four resulting relations, we conclude that the conditions for a, b, c, d are that

$$d = -a \qquad \text{and} \qquad bc = -1 - a^2$$

So there are infinitely many solutions, for example:

$$\begin{bmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}$$

Question 1

Expressing, for example, $x_1 = -9x_2 + 4x_3$, we can write the solution as

$$\begin{bmatrix} -9x_2 + 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

which is a plane defined by the two vectors specified in the above expression.

For the corresponding inhomogeneous equation,

$$\begin{bmatrix} 4 - 9x_2 + 4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

which is a parallel plane, shifted by vector $\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$.

NB: Certainly, any other variable can be expressed, for example $x_3 = x_1/4 + 9x_2/4$ and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_1/4 + 9x_2/4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1/4 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 9/4 \end{bmatrix}$$

which is the same plane, now defined by another two vectors.

Note that for the inhomogeneous equation, the plane shift is then also described by a different vector, $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ which, however, results in the same shifted plane as already obtained for the inhomogeneous case.

Question 2

Row reduction of the matrix gives

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the general solution to the homogeneous system is: $\quad \mathbf{x}_0 = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \qquad x_2 \in \mathbb{R}$

whereas row reduction for the augmented inhomogeneous matrix gives

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

from where, setting the free variable $x_2 = 0$, we take a particular solution $\mathbf{p} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$ and the complete solution is $\mathbf{x} = \mathbf{x}_0 + \mathbf{p}$.