Fundamentals of linear algebra

- Linear combinations
- Span
- Subspaces spanned by a set
- Linear dependence or independence

A *linear space* V is a non-empty set of objects, for which two operations are defined so that $\forall\,\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V$ and $\forall\, c,\ d \in \mathbb{R}$:

- addition $(\mathbf{u}+\mathbf{v}) \in V$
- multiplication by scalars $(c\,\mathbf{u}) \in V$

and these operations obey the following axioms:

(i) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iii) $\exists\,\mathbf{0} \quad : \quad \mathbf{u}+\mathbf{0} = \mathbf{u}$

(iv) $\exists\,(-\mathbf{u}) \quad : \quad \mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

(v) $1\cdot\mathbf{u} = \mathbf{u}$

(vi) $c\,(d\,\mathbf{u}) = (c\,d)\mathbf{u}$

(vii) $(c+d)\mathbf{u} = c\,\mathbf{u}+d\,\mathbf{u}$

(viii) $c\,(\mathbf{u}+\mathbf{v}) = c\,\mathbf{u}+c\,\mathbf{v}$

Consequences:

(ix) $\mathbf{0}$ is unique

(x) $(-\mathbf{u})$ is unique

(xi) $0\cdot\mathbf{u} = \mathbf{0}$

(xii) $(-\mathbf{u}) = (-1)\cdot\mathbf{u}$

(xiii) $c\cdot\mathbf{0} = \mathbf{0}$

(xiv) $\mathbf{u}-\mathbf{v} = \mathbf{u}+(-\mathbf{v})$

$(\mathbf{w} = \mathbf{u}-\mathbf{v}$ if $\mathbf{w}+\mathbf{v} = \mathbf{u})$

Revision: Subspaces

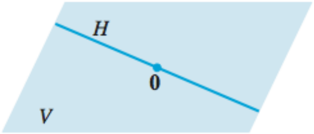
Definition: A **subspace** H of a linear space V is a subset of elements with the following properties:

- H is closed under addition: $\forall\,(\mathbf{u},\mathbf{v}) \in H,\ (\mathbf{u}+\mathbf{v}) \in H$
- H is closed under multiplication by scalars:
  $\forall\,\mathbf{u} \in H$ and $\forall\,c \in \mathbb{R}$, $c\,\mathbf{u} \in H$

Every subspace is a linear space and satisfies the axioms.

Property of any subspace: H includes the zero element of V

(so if a set does not include the zero element, then it is not a subspace)



Definition: Given a set of elements $\{\mathbf{v}_1,\, \mathbf{v}_2 \ldots\, \mathbf{v}_m\} \in V$
and given scalars $\{c_1,\, c_2 \ldots c_m\} \in \mathbb{R}$
(any real numbers including zero),

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_m\mathbf{v}_m$$

is called a *linear combination* of the set
$\{\mathbf{v}_1,\, \mathbf{v}_2 \ldots\, \mathbf{v}_m\}$ with coefficients $\{c_1,\, c_2 \ldots c_m\}$.

Example 1 (with vectors):

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\mathbf{y} = 2\mathbf{v}_1 + 3\mathbf{v}_2 = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

- **Example 2:** Given $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t^3$, and $\mathbf{p}_3(t) = 4 - t$:

  $$\mathbf{p}_4(t) = t^3 + t + 3$$

  is a linear combination of the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ because

  $$\mathbf{p}_4 = 7\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3$$

  whereas

  $$\mathbf{p}_5(t) = t^2 + 2$$

  is not a linear combination of these polynomials.

- **Example 3:** Given the interval $x \in [0,\, 2\pi]$, function

  $$f(x) = 2\sin x - 3\cos x$$

  is a linear combination of the functions $\{\sin x,\, \cos x\}$, whereas

  $$g(x) = 5\tan x + 4\cos x$$

  is not a linear combination of those functions.

If $\mathbf{A}$ is an $m \times n$ matrix with columns $\mathbf{a}_1 \ldots \mathbf{a}_n$ and if $\mathbf{x} \in \mathbb{R}^n$ then $\mathbf{Ax}$ is the linear combination of the columns of $\mathbf{A}$:

$$\mathbf{Ax} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \ldots + x_n\mathbf{a}_n$$

In general, $\mathbf{Ax} = \mathbf{b}$ can be interpreted as a linear combination

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

as well as a system of linear equations for the unknowns x_i.

This system can be solved by row-reducing the augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{array}\right]$$

Example: can $\mathbf{b}$ be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

That is, are there scalars x_1, x_2 such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$

Solution: Equivalent formulations of the problem:

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \qquad \Leftrightarrow$$

$$\left\{ \begin{array}{rcrcr} x_1 & + & 2x_2 & = & 7 \\ -2x_1 & + & 5x_2 & = & 4 \\ -5x_1 & + & 6x_2 & = & -3 \end{array} \right. \qquad \Leftrightarrow \qquad \begin{bmatrix} 1 & 2 \\ -2 & 5 \\ -5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

3 equations for 2 unknowns, so it might easily have no solution

The augmented matrix of the system is $\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right]$

Reduce to REF with $\mathsf{R}_2 \to \mathsf{R}_2 + 2\mathsf{R}_1$ and then $\mathsf{R}_3 \to \mathsf{R}_3 + 5\mathsf{R}_1$

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

So, there is a solution: $x_1 = 3$, $x_2 = 2$, which means

$$\mathbf{b} = 3\mathbf{a}_1 + 2\mathbf{a}_2$$

and thus $\mathbf{b}$ can be presented as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Note: vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{b}$ formed the the augmented matrix $\left[\mathbf{a}_1\ \mathbf{a}_2 \,|\, \mathbf{b}\right]$

Definition: For elements $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p \in V$, the set of all their linear combinations is denoted by $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p\}$

(it can be called a subset of V *spanned* by $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p$)

In other words, span is the collection of all elements produced as

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with arbitrary scalars c_1, c_2 ... c_p (possibly including zeros).

Vectors: for a given vector $\mathbf{b} \in \mathbb{R}^n$ and a set $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p \in \mathbb{R}^n$, the following statements are equivalent :

- $\mathbf{b} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p\}$
- $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p$
- Equation $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p \end{bmatrix} \mathbf{x} = \mathbf{b}$ has a solution

Example 1: check if $\mathbf{b} \in \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$$

Solution: row-reduce the augmented matrix $\left[\, \mathbf{a}_1 \; \mathbf{a}_2 \mid \mathbf{b} \,\right]$:

$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

The bottom row in the REF shows an inconsistency $0 \overset{!}{=} -2$, so there are no solutions and therefore $\mathbf{b} \notin \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$, that is, vector $\mathbf{b}$ is not a linear combination of vectors $\mathbf{a}_1$ and $\mathbf{a}_2$.

Example 2: Find h such that $\mathbf{y} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbb{R}^3$, if

$$\mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}; \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

Solution: vector $\mathbf{y} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{y}$

This vector equation corresponds to matrix equation

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix}$$

which is only consistent if $h = 5$, so then $\mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix}$.

Continue reduction towards REF, taking into account $h = 5$:

$$\begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which gives free variable x_3 and $\quad x_1 = 1 - 7x_3\,, \quad x_2 = -1 + 2x_3$

Setting the free variable $x_3 = 0$, we get $x_1 = 1$ and $x_2 = -1$.

Thus $\quad \mathbf{y} = 1\mathbf{v}_1 - 1\mathbf{v}_2 + 0\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2\,,$ which is easily checked:

$$\begin{bmatrix} -4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} - \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}.$$

Note: it is also possible to choose $x_3 \neq 0$, in which case the linear combination will involve all the three vectors.

Theorem:

For $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p \in V$, $\quad H = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p\}$ is a subspace of V.

H is called a subspace spanned (generated) by $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p\}$.

(proof)

(a) Any two elements in H can be written as

$\mathbf{u} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$ and $\mathbf{w} = s_1\mathbf{v}_1 + \cdots + s_p\mathbf{v}_p$

therefore their sum $\mathbf{u} + \mathbf{w} \in H$ because

$$\begin{aligned}\mathbf{u} + \mathbf{w} &= c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p + s_1\mathbf{v}_1 + \cdots + s_p\mathbf{v}_p \\ &= (c_1 + s_1)\mathbf{v}_1 + \cdots + (c_p + s_p)\mathbf{v}_p\end{aligned}$$

(b) For any $c \in \mathbb{R}$ element $c\,\mathbf{u} \in H$ because

$c\,\mathbf{u} = c(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = (cc_1)\mathbf{v}_1 + \cdots + (cc_p)\mathbf{v}_p$

Thus H is a subspace of V.

Needless to say, zero elements is in H: $\quad \mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_p$.

Example:

Let H be a set of all vectors of the form $\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix}$

where a, b are arbitrary scalars.

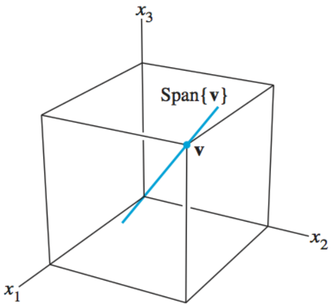
This can be rewritten as follows:

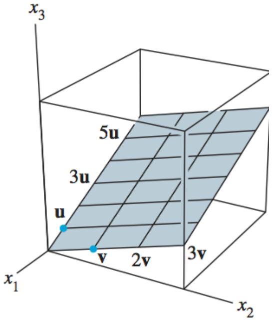
$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \equiv a\mathbf{v}_1 + b\mathbf{v}_2.$$

This rearrangement demonstrates that $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Therefore, H is a subspace of $\mathbb{R}^4$ generated by $\mathbf{v}_1$ and $\mathbf{v}_2$.

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$.
Then $\mathrm{Span}\{\mathbf{v}\} = c\,\mathbf{v}$ is the set of all scalar multiples of $\mathbf{v}$.
This can be visualised as the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$.





If $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$ are nonzero vectors with $\mathbf{u}$ not a multiple of $\mathbf{v}$, then $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\} = c_1\mathbf{u} + c_2\mathbf{v}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}, \mathbf{v}, \mathbf{0}$.

A set of elements $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n \in V$ is said to span V :

$$\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = V$$

if every element in V is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$.

In particular, the columns of $\mathbf{A}$ span $\mathbb{R}^m$ if every $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of $\mathbf{A}$.

For $m \times n$ matrix $\mathbf{A}$, the following statements are equivalent:

- The columns of $\mathbf{A}$ span $\mathbb{R}^m$
- $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution $\forall\, \mathbf{b} \in \mathbb{R}^m$
- Row-reduced $\mathbf{A}$ has a pivot position in every row

Example 1: check if these vectors span $\mathbb{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}.$$

Row-reduce $\begin{bmatrix}\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\end{bmatrix}$ and see if we get a pivot in every row.

$$\mathsf{R}_3 \to \mathsf{R}_3 + \mathsf{R}_1 \qquad\qquad \mathsf{R}_3 \to \mathsf{R}_3 - 2\mathsf{R}_2$$

$$\begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -2 \\ -1 & 7 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -2 \\ 0 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 3 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

So there is a pivot in every row and thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ span $\mathbb{R}^3$.

Any vector from $\mathbb{R}^3$ is a linear combination of these vectors.

Example 2: Check if the system $\mathbf{Ax} = \mathbf{b}$ is consistent for all $\mathbf{b}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Solution: $$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{array}\right] \qquad \begin{array}{l} \mathsf{R}_2 \to \mathsf{R}_2 + 4\mathsf{R}_1 \\ \mathsf{R}_3 \to \mathsf{R}_3 + 3\mathsf{R}_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{array}\right] \qquad \mathsf{R}_3 \to \mathsf{R}_3 - \frac{1}{2}\mathsf{R}_2$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & 3b_1 + \frac{1}{2}(-4b_1 - b_2) + b_3 \end{array}\right]$$

— only two pivots, therefore columns of $\mathbf{A}$ do not span $\mathbb{R}^3$.

So, the EF of the augmented matrix upon the reduction is:

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & b_1 - \frac{1}{2}b_2 + b_3 \end{array}\right]$$

The system is only consistent if: $b_1 - (1/2)b_2 + b_3 = 0$.

For consistency, the right-hand side must have the form

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ -b_1 + (1/2)b_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$$

i.e. $$\mathbf{b} \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} \right\}$$

This statement represents all the possible solutions for $\mathbf{Ax} = \mathbf{b}$.

So any vector spanned by the columns of $\mathbf{A}$ must have the form

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad \text{where} \qquad b_1 - \frac{1}{2}b_2 + b_3 = 0$$

In particular, each column of $\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$

satisfies the same constraint as $b_1 - b_2/2 + b_3 = 0$:

First column: $1 - (-4)/2 - 3 = 0$

Second column: $3 - 2/2 - 2 = 0$

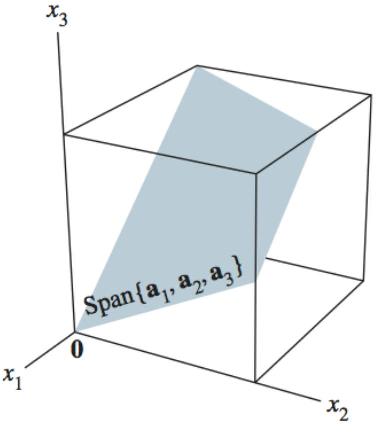
Third column: $4 - (-6)/2 - 7 = 0$

Geometrically, this relationship defines a plane:

$$x_1 - \frac{1}{2}x_2 + x_3 = 0$$

$$\mathrm{Span}\{\,\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3\,\} =$$

$$\mathrm{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}\right\}$$



The vector space spanned by the columns of $\mathbf{A}$ is generated by only two vectors (corresponding to REF $\mathbf{A}$ having only two pivots).

For this example, revisiting the equivalent statements reveals:

- " The columns of $\mathbf{A}$ span $\mathbb{R}^m$ "
  False: Columns of $\mathbf{A}$ only span a plane (not the entire $\mathbb{R}^3$)
- " $\mathbf{Ax} = \mathbf{b}$ has a solution $\forall\, \mathbf{b} \in \mathbb{R}^m$ "
  False: System only has a solution if $b_1 - b_2/2 + b_3 = 0$.
- " Row-reduced $\mathbf{A}$ has a pivot position in every row "
  False: REF $\mathbf{A}$ has only 2 pivots, not 3.

The columns of $\mathbf{A}$ span a plane defined by two vectors:

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

- A set $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_m\} \in V$ is *linearly independent* if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$$

has only the trivial solution (with all $c_i = 0$).

- A set $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_m\} \in V$ is *linearly dependent* if there are weights $c_1, c_2, \dots c_m$, not <u>all</u> equal to zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}.$$

The above relation is a linear dependence relation.

Linear (in)dependence is the property of a whole given set.

Not every element must be a linear combination of the other ones.

Linear dependence equation does not necessarily include all elements.

Example 1: Find if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

To do so, we need to solve the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augmented matrix can be reduced to REF as

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which indicates a nontrivial solution, so $\{\mathbf{v}_i\}$ is linearly dependent.

Now we can obtain an explicit form of the linear dependence

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$$

seeing that x_1, x_2 are basic, and x_3 is a free variable:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \left\{ \begin{array}{rcl} x_1 & = & 2x_3 \\ x_2 & = & -x_3 \\ \forall\, x_3 & \in & \mathbb{R} \end{array} \right.$$

Any non-zero value for x_3 yields a nontrivial solution; e.g. $x_3 = 1$:

$$2\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$$

This is a linear dependence equation for $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$.

Any other coefficients satisfying the above relations are suitable:

$$4\mathbf{v}_1 - 2\mathbf{v}_2 + 2\mathbf{v}_3 = \mathbf{0}$$
$$-2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$$
$$\dots$$

- **Example 2:** Given $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, and $\mathbf{p}_3(t) = 4 - t$, this polynomial set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\} \in \mathbb{P}$ is linearly dependent, because $\mathbf{p}_3 = 4\mathbf{p}_1 - \mathbf{p}_2$, that is, $4\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 = 0$

- **Example 3:** In $\mathcal{F}_{[0,\,2\pi]}$ (continuous functions on $0 \leqslant t \leqslant 2\pi$) the set $\{a \sin t,\, b \cos t\}$ is linearly independent:
  $c_1 \sin t + c_2 \cos t = 0$ only has the trivial solution, as there is no scalar c such that $\cos t = c \sin t \quad \forall\, t \in [0,\, 2\pi]$.

- **Example 4:** $\{\sin t \cos t,\, \sin 2t\} \in \mathcal{F}_{[0,\,\pi]}$ is linearly dependent, because $\sin 2t = 2 \sin t \cos t \;\; \forall\, t \in [0,\, \pi]$,
  so the linear dependence equation is $2 \sin t \cos t - \sin 2t = 0$.

- Zero element is linearly dependent as the equation $c \cdot \mathbf{0} = \mathbf{0}$ has infinite number of non-trivial solutions.

- One element $\{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$ (because $c\,\mathbf{v} = \mathbf{0}$ has only the trivial solution for $\mathbf{v} \neq \mathbf{0}$).

- The columns of matrix $\mathbf{A}$ are linearly independent if and only if the equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution.

  This directly follows from the fact that homogeneous equation $\mathbf{A}\mathbf{x} = \mathbf{0}$ is equivalent to $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0}$.

  Any linear dependence relation between the columns of $\mathbf{A}$ corresponds to a nontrivial solution of $\mathbf{A}\mathbf{x} = \mathbf{0}$.

Example 5:

Check if the columns of this matrix are linearly independent:

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & -2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 13 \end{bmatrix}$$

The EF shows pivots in every row, there are no free variables

So each column has a pivot, then $\mathbf{Ax} = \mathbf{0}$ only has the trivial solution, therefore the columns of $\mathbf{A}$ are linearly independent.

In other words, equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{0}$ only has a trivial solution, therefore vectors $\{\mathbf{a}_i\}$ are linearly independent.

Easy check for a set of two elements

For a set of two elements *only*, linear dependence is easy to check: It is sufficient to check if they are multiples of each other. Proof:

Suppose there are scalars $c \neq 0$ and d such that $c\mathbf{v} + d\mathbf{u} = \mathbf{0}$. Then $\mathbf{v} = (-d/c)\mathbf{u}$, implying them to be multiples of each other.

Vice versa, if $\mathbf{v} = k\mathbf{u}$, then $\mathbf{v} + (-k)\mathbf{u} = 0$, with at least one non-zero coefficient, so the two elements are linearly dependent.

Example 6: $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ — linearly dependent

Example 7: $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$ — linearly independent

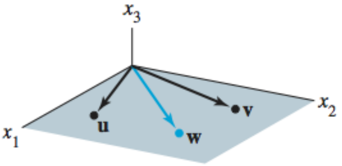
Example 8: $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}$ — linearly independent

(1) If $\mathbf{w} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly dependent set.

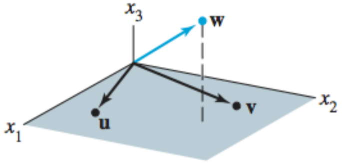
Proof: If $\mathbf{w} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$ then $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$, which can be rewritten as $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ with non-trivial $c_3 = -1$. Therefore, the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

(2) If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, but the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, then $\mathbf{w} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

Proof: If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent, then $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ with $c_3 \neq 0$ in this case (why?). Therefore $\mathbf{w} = -(c_1/c_3)\mathbf{u} - (c_2/c_3)\mathbf{v}$, implying $\mathbf{w} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.



Linearly dependent,
w in Span{**u**, **v**}



Linearly independent,
w *not* in Span{**u**, **v**}

Theorem:

- A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ of two or more elements is linearly dependent if and only if at least one of the elements in S is a linear combination of the other elements in S.
- If S is a linearly dependent set, then some $\mathbf{v}_j$ is a linear combination of the preceding elements $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_{j-1}$.

Proof follows the same logic as in the previous example with 3 elements.

Theorem:

- Any set $\{\mathbf{v}_1, \ldots \mathbf{v}_p\} \in \mathbb{R}^n$ is linearly dependent if $p > n$. That is, if a set contains more vectors than there are entries in each vector, then the set is linearly dependent.

Proof: Compose $\mathbf{A} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$, an $n \times p$ matrix. Then $\mathbf{Ax} = \mathbf{0}$ corresponds to n equations with p unknowns. If $p > n$, there are more variables than equations so there must be free variables and non-trivial solutions, so the columns of $\mathbf{A}$ are linearly dependent.

Example 9: $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$

must be linearly dependent because there are only two entries in each vector $(n = 2)$ but there are 3 vectors $(p = 3)$.

Indeed, if we compose the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$, upon reduction the corresponding matrix is

$$\begin{bmatrix} 2 & 4 & -2 \\ 1 & -1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The basic variables are x_1, x_2 and the free variable is x_3.

Thereby $x_1 = -x_3$ and $x_2 = x_3$

An explicit linear dependence equation is e.g.: $\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

Linear independence: example 10

Consider a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_6\}$ given by these columns:

$$\begin{bmatrix} 12 & 10 & -6 & -3 & 7 & 10 \\ -7 & -6 & 4 & 7 & -9 & 5 \\ 9 & 9 & -9 & -5 & 5 & -1 \\ -4 & -3 & 1 & 6 & -8 & 9 \\ 8 & 7 & -5 & -9 & 11 & -8 \end{bmatrix}$$

Upon row reduction, we find how $\mathbf{v}_i$ are related.

$$\to \begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The set is dependent, but we see pivots in columns 1, 2, 4, 6.

The REF form provides all the information on linear dependence:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns (1, 2, 4, 6) are linearly independent.

Therefore $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_4$, $\mathbf{v}_6$ are linearly independent.

The dependence coefficients appear in the non-pivot columns.

$$\mathbf{v}_3 = 2\mathbf{v}_1 - 3\mathbf{v}_2 \qquad \text{and} \qquad \mathbf{v}_5 = 2\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_4$$

Hence the examples of linear dependence equations:

$$2\mathbf{v}_1 - 3\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0} \qquad \text{and} \qquad 2\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_4 - \mathbf{v}_5 = \mathbf{0}$$

Linear independence: example 10

A long way to establish this relation runs by solving the system:

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2x_3 & - & 2x_5 \\ 3x_3 & + & 2x_5 \\ x_3 & & \\ & & x_5 \\ & & x_5 \\ & 0 & \end{bmatrix}$ can be written as:

$$\begin{cases} x_1 = -2x_3 - 2x_5 \\ x_2 = 3x_3 + 2x_5 \end{cases} \quad \text{and} \quad \begin{cases} x_4 = x_5 \\ x_6 = 0 \end{cases}$$

Using the solution in the form

$$\begin{cases} x_1 = -2x_3 - 2x_5 \\ x_2 = 3x_3 + 2x_5 \end{cases} \qquad \text{and} \qquad \begin{cases} x_4 = x_5 \\ x_6 = 0 \end{cases}$$

we can rewrite the vector equation as follows:

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + x_4\mathbf{v}_4 + x_5\mathbf{v}_5 + x_6\mathbf{v}_6 = \mathbf{0}$$

$$(-2x_3 - 2x_5)\mathbf{v}_1 + (3x_3 + 2x_5)\mathbf{v}_2 + x_3\mathbf{v}_3 + x_5\mathbf{v}_4 + x_5\mathbf{v}_5 + 0 \cdot \mathbf{v}_6 = \mathbf{0}$$

$$x_3(-2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3) + x_5(-2\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_4 + \mathbf{v}_5) = \mathbf{0}$$

The latter relation must hold true for any x_3 and x_5, therefore

$$\mathbf{v}_3 = 2\mathbf{v}_1 - 3\mathbf{v}_2 \qquad \text{and} \qquad \mathbf{v}_5 = 2\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_4$$

(as we have figured out already, analysing the REF matrix).

- A single element $\mathbf{v}$ is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$.

- A set of two non-zero elements $\{\mathbf{u}, \mathbf{v}\}$ is linearly dependent if and only if one is a multiple of the other.

- A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ of two or more elements is linearly dependent if and only if at least one of the elements in S is a linear combination of the other elements in S.

- If S is a linearly dependent set, then some $\mathbf{v}_j$ is a linear combination of the preceding elements $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_{j-1}$.

- If a set contains $\mathbf{0}$, then the set is linearly dependent.
  (suppose $\mathbf{v}_i = \mathbf{0}$, then $0\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 1\mathbf{v}_i + \ldots + 0\mathbf{v}_m = \mathbf{0}$ which demonstrates a linear dependence)

- Any set $\{\mathbf{v}_1, \ldots \mathbf{v}_p\} \in \mathbb{R}^n$ is linearly dependent if $p > n$.

Summary

- Linear combinations and span
- Subspaces spanned by a set
- Linear dependence or independence

Questions?

Class tests 2 are running this week at the tutorials

See you next week