University of Technology Sydney

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37233 Linear Algebra

Solutions 3

Question 1

Row reduction of the augmented matrix provides $\left[\begin{array}{cc|c} 2 & -2 & b_1 \\ 0 & 6 & b_1 - (3/2)\, b_2 \\ 0 & 0 & b_3 - 2b_1 \end{array}\right]$

so the system is only consistent if $b_3 - 2b_1 = 0$, which implies $\mathbf{b}$ such that

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ 2b_1 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{b} \in \mathrm{Span}\,\{\mathbf{v}_1,\, \mathbf{v}_2\} \quad \text{with} \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

which means $\mathbf{b}$ must belong to a plane in $\mathbb{R}^3$ that contains vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Question 2

Row-reducing the matrix formed from these vectors as its columns yields

$$\cdots \xrightarrow{\text{minimum}} \begin{bmatrix} 3 & 2 & 3 & 2 \\ 0 & -7/3 & 1 & -4/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{optional}} \begin{bmatrix} 1 & 0 & 9/7 & 0 \\ 0 & 1 & -3/7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

indicating a pivot in every row and therefore there is a solution for any right-hand side, so any vector in $\mathbb{R}^3$ is a linear combinations of vectors $\mathbf{v}_i$ and thus $\mathrm{Span}\{\mathbf{v}_i\} = \mathbb{R}^3$.

Question 3

To check for linear dependence, we should check of the corresponding homogeneous system has nontrivial. Row-reduction of the matrix $[\,\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\,]$ yields

$$\longrightarrow \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so there are non-trivial. We see three pivots and one free variable; thereby the first three vectors are linearly independent, and the REF form shows that the last vector is $\mathbf{v}_4 = 6\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3$ which can be rewritten as the linear dependence equation, e.g.

$$6\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 = \mathbf{0}$$

Question 4

For $\mathbf{A}$, regarding its columns in the reverse order, we immediately see that there will be four pivots in the REF, so the columns are linearly independent.

For $\mathbf{B}$, it is easy to see that e.g. the last column can be obtained as the second plus the third minus the first, so all the set of all these columns is linearly dependent.

For $\mathbf{C}$, like for $\mathbf{A}$, it can be immediately seen that the columns are linearly independent.