University of Technology Sydney

School of Mathematical and Physical Sciences

37233 Linear Algebra

Tutorial 3 — solution guide

Question 1

For $\mathbf{y}$ to be a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, the augmented matrix for the system $[\mathbf{v}_1\,\mathbf{v}_2\,|\,\mathbf{y}]$ must be consistent. Row reductions yields

$$\begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 28-2h \end{bmatrix}$$

so it must be that $h = 14$ and then $\mathbf{y}|_{h=14} = 8\mathbf{v}_1 - 3\mathbf{v}_2$.

Question 2

After swapping rows 1 and 2, then 2 and 3, the matrix $[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3]$ can be row-reduced to

$$\begin{bmatrix} 1 & -3 & -1 \\ 0 & 2 & -7 \\ 0 & 0 & 4 \end{bmatrix}$$

(and further to an identity matrix), so it has pivots in every row, so the system has a solution for any right-hand side, and therefore these three vectors span the entire $\mathbb{R}^3$.

Question 3

There are more columns than rows, so the columns must be linearly dependent: there may be no more than four pivots. In fact, row reduction for this matrix shows only three pivots:

$$\mathbf{A} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The REF allows to conclude that the set $\{\mathbf{a}_1,\, \mathbf{a}_2,\, \mathbf{a}_4\}$ can be used as a linearly independent set, and the other columns can be expressed as $\mathbf{a}_3 = 2\mathbf{a}_1$ and $\mathbf{a}_5 = -\mathbf{a}_1 + 5\mathbf{a}_2 + 2\mathbf{a}_4$, which can serve as the linear dependence equations.

Question 4

(i) There are only three vectors, so they cannot span $\mathbb{R}^4$.

(ii) There are four columns in $\mathbf{A}^\mathsf{T}$ so they can span $\mathbb{R}^3$ if there are at least three linearly independent columns. Row-reduction of $\mathbf{A}^\mathsf{T}$ results in

$$\begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

showing three pivots and therefore three linearly independent vectors. In other words, the system will have a solution for any right-hand side. So, the columns of $\mathbf{A}^\mathsf{T}$ span $\mathbb{R}^3$.

(iii) The columns of $\mathbf{A}$ are linearly independent as can be shown by row reduction to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The columns of $\mathbf{A}$ span a 3D space within $\mathbb{R}^4$.

(iv) As already found in (ii), the columns of $\mathbf{A}^\mathsf{T}$ are linearly dependent; there are only three linearly independent vectors within this set (let us denote it $\{\mathbf{v}_i\}$). As follows from the row-reduced matrix, a linear dependence relation can be written as $\mathbf{v}_3 = 4\mathbf{v}_1 - 2\mathbf{v}_2$.

(Question 5 omitted)

Question 6

(a) True by the equivalence of matrix form and vector form of the equation.

(b) False, as even if pivots are not found in every row, there are still vectors $\mathbf{b}$ such that the equation may have a solution.

(c) True, because then $\mathbf{b}$ is a linear combination of the columns of $\mathbf{A}$.

(d) True, because if it were consistent $\forall\, \mathbf{b} \in \mathbb{R}^n$, then the columns of $\mathbf{A}$ would have spanned the entire $\mathbb{R}^n$.

Question 7

Row reduction of the augmented matrix $[\mathbf{A} \,|\, \mathbf{b}]$ gives

$$\begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & (h-7) \end{bmatrix}$$

which is only consistent if $h = 7$.

There are only two pivots, so the columns of $\mathbf{A}$ do not span entire $\mathbb{R}^3$, but only a plane within $\mathbb{R}^3$. In case $h = 7$, vector $\mathbf{b}$ belongs to the same plane. Otherwise, vector $\mathbf{b}$ is not in the plane spanned by the columns of $\mathbf{A}$.