Fundamentals of linear algebra

- Brief revision: Span, linear (in)dependence
- Basis
- Dimension of a linear space / subspace
- Isomorphism of linear spaces

A *linear space* V is a non-empty set of objects, for which two operations are defined so that $\forall\, \mathbf{u},\ \mathbf{v},\ \mathbf{w} \in V$ and $\forall\, c,\ d \in \mathbb{R}$:

- addition $(\mathbf{u} + \mathbf{v}) \in V$
- multiplication by scalars $(c\,\mathbf{u}) \in V$

and these operations obey the following axioms:

(i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(iii) $\exists\, \mathbf{0} \quad : \quad \mathbf{u} + \mathbf{0} = \mathbf{u}$

(iv) $\exists\, (-\mathbf{u}) \quad : \quad \mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(v) $1 \cdot \mathbf{u} = \mathbf{u}$

(vi) $c\,(d\,\mathbf{u}) = (c\,d)\mathbf{u}$

(vii) $(c + d)\mathbf{u} = c\,\mathbf{u} + d\,\mathbf{u}$

(viii) $c\,(\mathbf{u} + \mathbf{v}) = c\,\mathbf{u} + c\,\mathbf{v}$

Consequences:

(ix) $\mathbf{0}$ is unique

(x) $(-\mathbf{u})$ is unique

(xi) $0 \cdot \mathbf{u} = \mathbf{0}$

(xii) $(-\mathbf{u}) = (-1) \cdot \mathbf{u}$

(xiii) $c \cdot \mathbf{0} = \mathbf{0}$

(xiv) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$

$(\mathbf{w} = \mathbf{u} - \mathbf{v}$ if $\mathbf{w} + \mathbf{v} = \mathbf{u})$

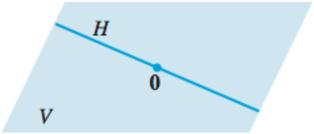
Definition: A **subspace** H of a linear space V is a subset of elements with the following properties:

- H is closed under addition: $\forall\,(\mathbf{u},\mathbf{v}) \in H,\ (\mathbf{u}+\mathbf{v}) \in H$
- H is closed under multiplication by scalars: $\forall\,\mathbf{u} \in H$ and $\forall\,c \in \mathbb{R},\ c\,\mathbf{u} \in H$

Every subspace is a linear space and satisfies the axioms.

Property of any subspace: H includes the zero element of V

(so if a set does not include the zero element, then it is not a subspace)



Definition: $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_n\mathbf{v}_n, \quad \mathbf{v}_i \in V, \quad c_i \in \mathbb{R}$
is called a *linear combination* of the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$
with coefficients $\{c_1, c_2 \ldots c_n\}$.

Definition: For elements $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p \in V$, the set of all their linear combinations is denoted by $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$ and it is a subspace of V *spanned* (generated) by $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p$.

Definition: A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ is *linearly independent* if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$$

has only the trivial solution (with all $c_i = 0$).

- If S is a linearly dependent set, then some $\mathbf{v}_j$ is a linear combination of the preceding elements $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_{j-1}$.
- If a set contains $\mathbf{0}$, then this set is linearly dependent.
- Vector set $\{\mathbf{v}_1, \ldots \mathbf{v}_p\} \in \mathbb{R}^n$ is linearly dependent if $p > n$.

Definition: Set $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_p\} \in V$ is a **basis** for subspace H

if: (i) $\mathcal{B}$ is a linearly independent set,

and (ii) $H = \text{Span}\{\mathbf{b}_1, \ldots \mathbf{b}_p\}$

Note: This definition also applies for $H = V$.

Theorem: If a linear space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ with n elements, then any set in V containing more than n elements must be linearly dependent.

Theorem: If a linear space V has a basis of n elements, then every basis of V must consist of exactly n elements.

Example 1: Set $\mathcal{P} = \{1,\, t,\, t^2,\, \ldots\ t^n\}$ is a basis for $\mathbb{P}_n$.
This basis is called the *standard basis* for polynomial space $\mathbb{P}_n$.

Proof: It is obvious that any polynomial of a degree up to n, can be written as a linear combination of the elements of $\mathcal{P}$. Therefore, set $\mathcal{P}$ spans the polynomial space $\mathbb{P}_n$.

To check $\mathcal{S}$ for linear independence, consider if

$$c_0 \cdot 1 + c_1 t + c_2 t^2 + \ldots + c_n t^n = 0 \qquad \forall\, t$$

However a polynomial of degree n has at most n roots, which means that, in general, the above relation can only be satisfied for not more than some n specific values of t, but not for any t.

Therefore the above relation can only be satisfied for all t if all $c_i = 0$, thus the set $\mathcal{P}$ is linearly independent.

Example 2: For any vector in $\mathbb{R}^2$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are the *standard basis* in $\mathbb{R}^2$

Similarly in $\mathbb{R}^3$: $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$; the standard basis is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Generally in $\mathbb{R}^n$ $\mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$ with $\mathbf{e}_n : \begin{cases} e_n^{(i=n)} = 1 \\ e_n^{(i\neq n)} = 0 \end{cases}$

(columns of the corresponding identity matrix)

Example 3: Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ if

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

Solution: Check that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ <u>spans $\mathbb{R}^3$</u>:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b} \qquad \text{is consistent} \quad \forall\, \mathbf{b} \in \mathbb{R}^3$$

and that this set is <u>linearly independent</u>:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \qquad \text{only if} \qquad c_i = 0 \;\forall\, i$$

(1) Spanning: solution for any 'right-hand' side (inhomogeneous eq.);

(2) Linear independence: only the trivial solution (homogeneous eq.).

Example 3: Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ if

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

So: we are checking if the set spans $\mathbb{R}^3$ **and** is linearly independent:

(1) Solution for any 'right-hand' side (for inhomogeneous eq.);

(2) Only the trivial solution (for homogeneous eq.).

Considering the equations in matrix form, this implies having:

(1) pivots in every row, and (2) pivots in every column.

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 6 & -2 & 5 \end{bmatrix} \to \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Pivots in every column and every row: $\{\mathbf{v}_i\}$ is a basis for $\mathbb{R}^3$.

Example 4: Consider $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$$

It is easy to see that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, therefore $\forall\, \mathbf{u} \in H$

$$\begin{aligned} \mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(5\mathbf{v}_1 + 3\mathbf{v}_2) \\ &= (c_1 + 5c_3)\mathbf{v}_1 + (c_2 + 3c_3)\mathbf{v}_2. \end{aligned}$$

Thus $\mathbf{u} \in \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and so H is identical to $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
In other words, it turns out $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.
Clearly, every vector in $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ belongs to H because

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0\mathbf{v}_3.$$

Then, we see that $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent ($\mathbf{v}_1 \neq c\mathbf{v}_2$).
Therefore, we can conclude that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for H.

Example 4: Consider $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$$

Note: We have written that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$.

However, equivalently $\mathbf{v}_1 = -\frac{3}{5}\mathbf{v}_2 + \frac{1}{5}\mathbf{v}_3$, or $\mathbf{v}_2 = \frac{1}{3}\mathbf{v}_3 - \frac{5}{3}\mathbf{v}_1$.

Thus $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathrm{Span}\{\mathbf{v}_2, \mathbf{v}_3\} = \mathrm{Span}\{\mathbf{v}_3, \mathbf{v}_1\}$.

The pair $\mathbf{v}_2$, $\mathbf{v}_3$ is linearly independent, and $\mathbf{v}_3$, $\mathbf{v}_1$ as well.

Therefore, $\{\mathbf{v}_2, \mathbf{v}_3\}$ is also a basis for H, and so is $\{\mathbf{v}_2, \mathbf{v}_3\}$.

Any two of these three vectors make a basis for H.

- A basis is an "efficient" spanning set that contains only "necessary" elements.
- A basis can be constructed from a spanning set by discarding unnecessary elements.

Theorem (the spanning set theorem):

Let $\mathcal{S} = \{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ be a set in V, and $H = \mathrm{Span}\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$.

(a) If one of the elements in $\mathcal{S}$, say $\mathbf{v}_i$, is a linear combination of the other elements in $\mathcal{S}$, then the set formed from $\mathcal{S}$ by removing $\mathbf{v}_i$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, some subset of $\mathcal{S}$ is a basis for H.

Spanning set theorem (proof)

(a): Suppose $\mathbf{v}_p = a_1\mathbf{v}_1 + \ldots + a_{p-1}\mathbf{v}_{p-1}$. Then $\forall\, \mathbf{x} \in H$,

$$\begin{aligned}\mathbf{x} &= c_1\mathbf{v}_1 + \ldots + c_{p-1}\mathbf{v}_{p-1} + c_p\mathbf{v}_p \\ &= c_1\mathbf{v}_1 + \ldots + c_{p-1}\mathbf{v}_{p-1} + c_p(a_1\mathbf{v}_1 + \ldots + a_{p-1}\mathbf{v}_{p-1}) \\ &= (c_1 + c_pa_1)\mathbf{v}_1 + \ldots + (c_{p-1} + c_pa_{p-1})\mathbf{v}_{p-1}\end{aligned}$$

Thus $\{\mathbf{v}_1, \ldots \mathbf{v}_{p-1}\}$ spans H, because this holds $\forall\, \mathbf{x} \in H$.

(b): If $\mathcal{S}$ is linearly independent, then it is already a basis for H.

If not, then one of the elements can be removed (see part a). We can continue to remove elements until the remaining set is linearly independent and hence is a basis for H.

If the set is eventually reduced to one element, that element will be non-zero because $H \neq \{\mathbf{0}\}$. Since a single non-zero element $\mathbf{v}$ is linearly independent, it will be a basis for H.

Notes:

- A basis is the smallest possible spanning set.
- A basis is the largest possible linearly independent set.
- If $\mathcal{S}$ is a basis for V and is enlarged by one element $\mathbf{w} \in V$ then the enlarged set cannot be linearly independent, because $\mathcal{S}$ spans V so $\mathbf{w}$ is a linear combination of the elements of $\mathcal{S}$.
- If $\mathcal{S}$ is a basis for V and if $\mathcal{S}$ is made smaller by one element $\mathbf{u} \in V$ then the reduced set cannot serve as a basis, because it will not span V anymore.

Any smaller linearly independent set can be enlarged to form a basis, but further enlargement destroys the linear independence.

Any excessively large spanning set can be reduced to a basis, but further shrinking destroys the spanning property.

Spanning set theorem (examples)

A linearly independent set which does **not** span $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}.$$

A basis for $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}.$$

A set that spans $\mathbb{R}^3$ but is linearly dependent:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}.$$

(Any) 3 linearly independent vectors form a basis for $\mathbb{R}^3$.

2 (or less) are not sufficient, while 4 (or more) are too many.

Definition: If V is spanned by a finite set, then V is called a *finite-dimensional* space, and the dimension of V, written as $\dim V = n$, is the number n of elements in a basis for V.

The dimension of the $\{\mathbf{0}\}$ linear space is defined to be zero.

If V is not spanned by a finite set then V is *infinite-dimensional*.

Warning: for a vector subspace, dimension is the number of elements in a basis, **not** a number of entries in each vector.

Theorem: (the basis theorem)

Let V be a n-dimensional linear space, $n \geqslant 1$. Then:

- any linearly independent set of n elements in V is a basis for V;
- any set of n elements that spans V is a basis for V.

Example: The basis for $\mathbb{R}^n$ contains n vectors, so $\dim \mathbb{R}^n = n$.

Example: The standard polynomial basis is $\{1, t, t^2, t^3, \ldots\}$,

so, e.g. $\dim \mathbb{P}_2 = 3$; $\dim \mathbb{P}_0 = 1$; and in general, $\dim \mathbb{P}_n = n + 1$.

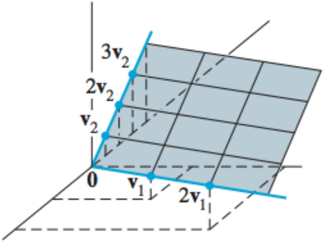
The space $\mathbb{P}$ of all polynomials is infinite-dimensional.

Example:

Let $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, with

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for H, since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. So $\dim H = 2$.



Theorem: Let H be a subspace of a finite-dimensional space V. Then any linearly independent set in H can be expanded to a basis for H, and $\dim H \leqslant \dim V$.

Proof: If $H = \{\mathbf{0}\}$, then certainly $\dim H = 0 \leqslant \dim V$.

Otherwise let $S = \{\mathbf{u}_1 \ldots \mathbf{u}_k\}$ be a linearly independent set in H.

If S spans H then S is a basis for H.

Otherwise there is a some $\mathbf{u}_{k+1} \in H$ which is not in span of S.

Then the set $S' = \{\mathbf{u}_1, \ldots \mathbf{u}_k, \mathbf{u}_{k+1}\}$ is linearly independent.

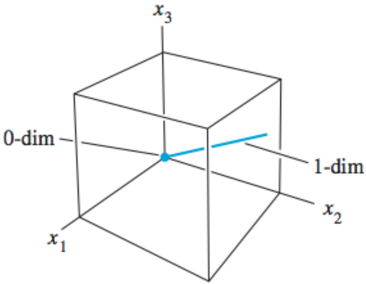
As long as the new set S' does not span subspace H we can continue to add linearly independent elements, expanding S to a larger linearly independent set in H.

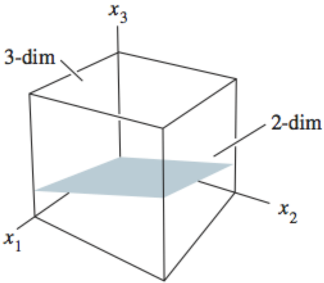
But the number of elements can never exceed $\dim V$, which is a number of linearly independent elements in the entire space V.

Eventually the expanded set S will span H and $\dim H \leqslant \dim V$.

Example: Various subspaces of $\mathbb{R}^3$

- *0*-dimensional subspace: only the zero subspace.
- *1*-dimensional subspaces: any subspace spanned by a single nonzero vector — these define lines through the origin.
- *2*-dimensional subspaces: any subspace spanned by two linearly independent vectors — planes through the origin.
- *3*-dimensional subspace: only $\mathbb{R}^3$ itself — any three linearly independent vectors in $\mathbb{R}^3$ span the entire $\mathbb{R}^3$.





Example: Find the dimension of a subspace defined as

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} \right\} \qquad a, b, c, d \in \mathbb{R}.$$

Solution: Decomposing these vectors, we have

$$\begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} = a \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

Therefore H is the set of all linear combination of vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}.$$

Example (a vector subspace)

By analysing these vectors, we notice the following

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}.$$

- $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$
- $\mathbf{v}_3$ is a multiple of $\mathbf{v}_2$ (since $\mathbf{v}_3 = -2\mathbf{v}_2$). By spanning set theorem, if we discard $\mathbf{v}_3$, the remaining set still spans H
- $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ (why?)

So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a linearly independent set and a basis for H.

Thus: $\dim H = 3$.

Alternatively, we could make a row-reduction of the corresponding matrix, to discover that there are three pivots only.

Example: Find the dimension for a set of all polynomials $\{q(t)\}$ of $\mathbb{P}_2$, such that $q(t) = 0$ for $t = 1$.

A general form for a polynomial of $\mathbb{P}_2$ is: $p(t) = c_0 + c_1 t + c_2 t^2$.

But $\forall\, q(t)$: $\quad q(1) = c_0 + c_1 \cdot 1 + c_2 \cdot 1^2 = c_0 + c_1 + c_2 = 0$

from where we can generally express e.g. $\quad c_0 = -c_1 - c_2$:

$$q(t) = (-c_1 - c_2) + c_1 t + c_2 t^2 = c_1(t-1) + c_2(t^2 - 1)$$

So any $q(t)$ is a linear combination of two polynomials

$$q_1(t) = t - 1 \qquad \text{and} \qquad q_2(t) = t^2 - 1$$

which are linearly independent. So $\{q_1, q_2\}$ is a basis for $\{q(t)\}$.

Therefore $\dim\{q(t)\} = 2$.

NB: The above basis is not unique. Expressing c_2 via c_0 and c_1, or c_1 via c_0 and c_2, we obtain other pairs of basis polynomials.

Definition: A *one-to-one correspondence* is said to be established between linear spaces V and W if every element $\mathbf{v} \in V$ is mapped to element $\mathbf{w} \in W$ so that each $\mathbf{w} \in W$ is mapped by only one element $\mathbf{v} \in V$. One of the usual notations is: $\mathbf{v} \leftrightarrow \mathbf{w}$.

Definition: Linear spaces V and W are called *isomorphic* if there is a one-to-one correspondence such that:

$(\mathbf{v}_1 + \mathbf{v}_2) \leftrightarrow (\mathbf{w}_1 + \mathbf{w}_2)$ if $\mathbf{v}_i \leftrightarrow \mathbf{w}_i$

and $c\,\mathbf{v} \leftrightarrow c\,\mathbf{w}$ if $\mathbf{v} \leftrightarrow \mathbf{w}$ (with the same c)

Theorem: Linear spaces V and W are isomorphic if and only if $\dim V = \dim W$.

Notation and terminology may be very different in V and W, however as linear spaces they have identical structure.

Example:

Set $\mathcal{P} = \{1, t, t^2, t^3\}$ is the standard basis in the space $\mathbb{P}_3$.

Thus $\dim \mathbb{P}_3 = \dim \mathbb{R}^4$, therefore $\mathbb{P}_3$ and $\mathbb{R}^4$ are isomorphic.

A one-to-one correspondence between a typical element of $\mathbb{P}_3$

$$\mathbf{p} = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

and a typical vector of $\mathbb{R}^4$ can be established:

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{bmatrix} \quad \text{etc.}$$

Any linear operations in $\mathbb{P}_3$ correspond to those in $\mathbb{R}^4$.

- Polynomial space $\mathbb{P}_n$ is isomorphic to vector space $\mathbb{R}^{(n+1)}$
- A space of all arrows defined in a geometric space with n independent directions is isomorphic to vector space $\mathbb{R}^n$
- A space of all $m \times n$ matrices $\mathbb{M}_n^m$ of real numbers is isomorphic to polynomial space $\mathbb{P}_{(m \cdot n - 1)}$ and to $\mathbb{R}^{(m \cdot n)}$
- A space of all functions $f(t) = a \sin t + b \cos t$ where $a,\, b \in \mathbb{R}$, is isomorphic to $\mathbb{R}^2$ as well as to $\mathbb{P}_1$

Establishing an isomorphism to an appropriate vector space is highly useful in the analysis of linear spaces, since vector spaces can be analysed with the necessary matrix equations.

Example: Check if these polynomials are linearly dependent:

$$\mathbf{p}_1 = 1 + 2t^2, \qquad \mathbf{p}_2 = 4 + t + 5t^2, \qquad \mathbf{p}_3 = 3 + 2t$$

Solution: These $\mathbf{p}_i \in \mathbb{P}_2$ which is isomorphic to $\mathbb{R}^3$:

$$\mathbf{p}_1 \leftrightarrow \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \mathbf{p}_2 \leftrightarrow \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \qquad \mathbf{p}_3 \leftrightarrow \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

Check for linear dependence of these vectors by row reduction:

$$\begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & -3 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

No pivot in the third column so the columns are linearly dependent.

So x_3 is a free variable, and then $x_2 = -2x_3$ and $x_1 = 5x_3$:

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Choosing e.g. $x_3 = 1$, we have $x_1 = 5$ and $x_2 = -2$:

$$5\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \mathbf{0}$$

Coming back to the polynomials, the corresponding relation is:

$5\mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0}$ which can be confirmed:

$$5(1 + 2t^2) - 2(4 + t + 5t^2) + (3 + 2t) = 0 \quad \forall\, t$$

Summary

- Basis

  (minimal spanning set, maximal linearly independent set)

- Dimension

  (number of elements in a basis)

- Isomorphism

  (equivalence between linear spaces)

Questions?

Class tests 3 are running this week at the tutorials

See you next week