University of Technology Sydney

School of Mathematical and Physical Sciences

37233 Linear Algebra

Solutions 4

Question 1

There are 5 vectors which belong to $\mathbb{R}^3$ so they must be linearly dependent. To find out their dependence, we row-reduce the matrix, obtaining

$$\begin{bmatrix} 1 & -2 & 7 & -4 & -6 \\ -2 & -1 & -9 & 8 & 7 \\ 1 & 13 & -8 & -4 & 9 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 & -4 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}$$

There are three pivots, so three linearly independent vectors can be found in this set. REF shows that $\mathbf{v}_4 = -4\mathbf{v}_1$ and $\mathbf{v}_5 = \mathbf{v}_1 - \mathbf{v}_3$.

(a) Thus there are 5 bases formed from the given vectors:

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_5\}$, $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_5\}$, $\{\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5\}$.

(b) $\mathbf{v}_3 = \Big|_{\{1,2,3\}} \mathbf{v}_3 = \Big|_{\{1,2,5\}} \mathbf{v}_1 - \mathbf{v}_5 = \Big|_{\{2,3,4\}} \mathbf{v}_3 = \Big|_{\{2,3,5\}} \mathbf{v}_3 = \Big|_{\{2,4,5\}} -\frac{1}{4}\mathbf{v}_4 - \mathbf{v}_5$

Question 2

(a) The set $\{\mathbf{E}_i\}$ is linearly independent if the corresponding linear dependence equation

$$c_1\mathbf{E}_1 + c_2\mathbf{E}_2 + c_3\mathbf{E}_3 + c_4\mathbf{E}_4 + c_5\mathbf{E}_5 + c_6\mathbf{E}_6 = \mathbf{0}_\mathbf{2}^\mathbf{3}$$

only has a trivial solution (where $\mathbf{0}_\mathbf{2}^\mathbf{3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the zero element of $\mathbb{M}_2^3$). Indeed:

$$c_1\mathbf{E}_1 + c_2\mathbf{E}_2 + c_3\mathbf{E}_3 + c_4\mathbf{E}_4 + c_5\mathbf{E}_5 + c_6\mathbf{E}_6 = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has the only solution that all $c_i = 0$. Therefore, the set is linearly independent.

(b) Any element of $\mathbb{M}_2^3$ can be represented as

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} = a_1\mathbf{E}_1 + a_2\mathbf{E}_2 + a_3\mathbf{E}_3 + a_4\mathbf{E}_4 + a_5\mathbf{E}_5 + a_6\mathbf{E}_6$$

which is a linear combination of $\{\mathbf{E}_i\}$ set.

(c) The set $\{\mathbf{E}_i\}$ spans $\mathbb{M}_2^3$ and is linearly independent, therefore it is a basis for $\mathbb{M}_2^3$.

Question 3

Expand vectors of H in terms of a linear combination with a, b, c coefficients:

$$H = \left\{ a \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 0 \\ 4 \\ 1 \\ 9 \end{bmatrix} + c \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -5 \\ -2 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Thus H is the set of all linear combination of the above vectors.

All the above vectors belong to $\mathbb{R}^4$. Clearly, addition of any vectors from H results in a vector that can be represented in the same form, so also belongs to H. Likewise, multiplication of any such vector by a scalar results in a vector that can be represented in the same form, so it also belongs to H. Therefore H is a subspace of $\mathbb{R}^4$.

To find how many of these vectors are linearly independent, row-reduce the matrix:

$$\begin{bmatrix} 1 & 0 & 3 & -5 \\ 2 & 4 & 2 & -2 \\ 0 & 1 & -1 & 2 \\ 3 & 9 & 0 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & -5 \\ 0 & 4 & -4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are two pivots, so only two vectors are linearly independent.

For example, this subspace can be represented as (using the first two columns as a basis)

$$H = \left\{ a \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + b \begin{bmatrix} 0 \\ 4 \\ 1 \\ 9 \end{bmatrix} \right\}.$$

Question 4

(a) Any non-zero element can serve as a basis for this linear space. The zero element for this space is number 1, so any $x \neq 1$ is a basis, for example, number $x = 2$.

(b) The basis contains one element, so the dimension of this linear space is equal to 1.