University of Technology Sydney

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37233 Linear Algebra

Tutorial 4 — solution guide

Question 1

(a) No, there are three vectors whereas only two are required to form a basis for $\mathbb{R}^2$. However, any two vectors from this set are linearly independent, so any pair of them can form a basis, hence there are three versions possible from this set.

(b) This is done with row-reduction of the augmented matrices:

$$[\mathbf{v}_1\ \mathbf{v}_2\,|\,\mathbf{x}] = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -8 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \end{bmatrix} \Rightarrow \mathbf{x} = 5\mathbf{v}_1 - 2\mathbf{v}_2$$

$$[\mathbf{v}_2\ \mathbf{v}_3\,|\,\mathbf{x}] = \begin{bmatrix} 2 & -3 & 1 \\ -8 & 7 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow \mathbf{x} = -\mathbf{v}_2 - \mathbf{v}_3$$

$$[\mathbf{v}_3\ \mathbf{v}_1\,|\,\mathbf{x}] = \begin{bmatrix} -3 & 1 & 1 \\ 7 & -3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -5 \end{bmatrix} \Rightarrow \mathbf{x} = -2\mathbf{v}_3 - 5\mathbf{v}_1$$

Question 2

(a) The two vectors are not multiples of each other so they are linearly independent, and thus they form a basis for H (can be also demonstrated explicitly with row reduction).

(b) Row-reducing the augmented matrices with $\mathbf{x}$ and $\mathbf{y}$ we get

$$\rightarrow \begin{bmatrix} 2 & 2 & 8 \\ 0 & -5 & -10 \\ 0 & -6 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathbf{x} \in H$$

$$\rightarrow \begin{bmatrix} 2 & 2 & 12 \\ 0 & -5 & -10 \\ 0 & 0 & -22 \end{bmatrix} : \text{ inconsistency } \Rightarrow \mathbf{y} \notin H$$

Question 3

Row-reducing the corresponding matrix we get

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which shows two pivots, so there are two linearly independent vectors which span H.

(a) The first two vectors can serve as a basis for H, and $\dim H = 2$.

(b) Each pair within this set is a linearly independent pair (observe that all these vectors are not multiples of each other). Thus, any pair from this set is a basis.

Question 4

A general form for a polynomial of $\mathbb{P}_3$ is $p(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3$.

Applying the condition which defines subset $\{q(t)\}$, to this form, we obtain

$$q(0) = c_0 + c_1 \cdot 0 + c_2 \cdot 0^2 + c_3 \cdot 0^3 = c_0$$

and thus $c_0 = 0$, and thereby any polynomials from $\{q(t)\}$ are: $q(t) = c_1 t + c_2 t^2 + c_3 t^3$.

So any such $q(t)$ is a linear combination of $q_1(t) = t$, $q_2(t) = t^2$, and $q_3(t) = t^3$. These three polynomials are linearly independent and therefore form a basis for $\{q(t)\}$.

Thus $\dim\{q(t)\} = 3$.

Question 5

Considering four standard matrices from $\mathbb{M}_2^2$ and writing their linear combination:

$$c_1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$$

we can see that they span $\mathbb{M}_2^2$ (any matrix can be represented as their linear combination), and they are linearly independent (there is only a trivial solution to result in the zero matrix).

Therefore these standard matrices form a basis for $\mathbb{M}_2^2$, and $\dim \mathbb{M}_2^2 = 4$.

Then any four linearly independent matrices form another basis for $\mathbb{M}_2^2$. We can check that the four W_i matrices are linearly independent: the equation

$$c_1\mathbf{W}_1 + c_2\mathbf{W}_2 + c_3\mathbf{W}_3 + c_4\mathbf{W}_4 = \begin{bmatrix} c_1 + c_2 + c_3 + c_4 & 3c_1 + 3c_2 \\ 2c_1 + 2c_2 + 2c_3 & 4c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

has the only solution that all $c_i = 0$ (as can be seen consecutively from c_1 to c_4).

Thereby, the set $\{W_i\}$ is a basis for $\mathbb{M}_2^2$.