University of Technology Sydney

School of Mathematical and Physical Sciences

37233 Linear Algebra

Exercises 5

Question 1

Consider vectors $\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$.

(a) Figure out if these vectors form a basis $\mathcal{B}$ for $\mathbb{R}^3$.

(b) Find $\mathcal{B}$-coordinates $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$.

(c) Find $\mathbf{y}$ which has $\mathcal{B}$-coordinates $[\mathbf{y}]_{\mathcal{B}} = \begin{pmatrix} 16 \\ 5 \\ 1 \end{pmatrix}$.

Question 2

Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ are bases for $\mathbb{R}^3$, even though we do not know the coordinates of all those vectors relative to the standard basis.

However, we know that $\mathbf{b}_1 = \mathbf{c}_1 - 2\mathbf{c}_2 + 2\mathbf{c}_3$, $\mathbf{b}_2 = 2\mathbf{c}_1 - 3\mathbf{c}_2 + 4\mathbf{c}_3$, and $\mathbf{b}_3 = 3\mathbf{c}_1 - 4\mathbf{c}_2 + 7\mathbf{c}_3$.

(a) Find $[\mathbf{x}]_{\mathcal{C}}$ given that $\mathbf{x} = -6\mathbf{b}_1 + 5\mathbf{b}_2 - \mathbf{b}_3$.

(b) Find $[\mathbf{y}]_{\mathcal{B}}$ given that $\mathbf{y} = \mathbf{c}_1 - 3\mathbf{c}_2 + 2\mathbf{c}_3$.

Question 3

Let

$$\mathbf{B} = \begin{bmatrix} 6 & 0 & 0 & 4 \\ 4 & 2 & 0 & 0 \\ 0 & -8 & 4 & 0 \\ 0 & 0 & 2 & -6 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 5 & -5 & 5 \\ 1 & 3 & -1 & 1 \\ 6 & -6 & -2 & 6 \\ 4 & -4 & 4 & -2 \end{bmatrix}$$

(a) Check if the columns of $\mathbf{B}$ form a basis $\mathcal{B}$ for $\mathbb{R}^4$, and if the columns of $\mathbf{C}$ form another basis $\mathcal{C}$ for $\mathbb{R}^4$.

(b) Find $\mathcal{B}$-coordinates of $\mathbf{x} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}$.

(c) Find the change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.

(d) Using the above matrix, find $[\mathbf{x}]_{\mathcal{C}}$.

Question 4

Hermite polynomials are used in mathematical physics and probability theory.

The first four of these polynomials are: $h_1 = 1$, $h_2 = 2t$, $h_3 = 4t^2 - 2$, and $h_4 = 8t^3 - 12t$.

(a) Show that these polynomials form a basis $\mathcal{H}$ for $\mathbb{P}^3$.

(b) Find the coordinates of a polynomial $p = t^3 + t^2 + t + 1$ in the basis $\mathcal{H}$.

(c) Write down polynomial q with $\mathcal{H}$-coordinates $[q]_{\mathcal{H}} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(d) Take the following set $\{q_i\}$ of polynomials in $\mathbb{P}^3$:

$q_1 = 1, \quad q_2 = t + 1, \quad q_3 = t^2 + t + 1, \quad \text{and} \quad q_4 = t^3 + t^2 + t + 1.$

(i) Determine if set $\{q_i\}$ forms a basis $\mathcal{Q}$ in $\mathbb{P}^3$.

(ii) Obtain the change of coordinates matrix $\mathbf{P}_{\mathcal{Q} \leftarrow \mathcal{H}}$.

(iii) Calculate the coordinates of $r = t \cdot (1 + 2t + 3t^2)$ relative to $\mathcal{H}$.

(iv) Use the change of coordinates matrix found in (b) to calculate $[r]_{\mathcal{Q}}$.