Fundamentals of linear algebra

- Revision: Basis, dimension, isomorphism
- Coordinate systems
- Change of basis

Definition: Set $\mathcal{B} = \{\mathbf{b}_1, \dots \mathbf{b}_p\} \in V$ is a **basis** for V if
(i) $\mathcal{B}$ is a linearly independent set, and
(ii) $V = \mathrm{Span}\{\mathbf{b}_1, \dots \mathbf{b}_p\}$

Notes:

- A basis is the smallest possible spanning set.
- A basis is the largest possible linearly independent set.
- If $\mathcal{S}$ is a basis for V and is enlarged by one element $\mathbf{w} \in V$ then the enlarged set cannot be linearly independent, because $\mathcal{S}$ spans V so $\mathbf{w}$ is a linear combination of the elements of $\mathcal{S}$.
- If $\mathcal{S}$ is a basis for V and if $\mathcal{S}$ is made smaller by one element $\mathbf{u} \in V$ then the reduced set cannot serve as a basis, because it will not span V anymore.

Definition: If V is spanned by a finite set, then V is called a *finite-dimensional* space, and the dimension of V, written as $\dim V = n$, is the number n of elements in a basis for V.

Warning: for a vector space, dimension is the number of elements in a basis, **not** a number of entries in each vector.

Theorem: Let V be a p-dimensional linear space, $p \geqslant 1$. Then:

- any linearly independent set of p elements in V is a basis for V;
- any set of p elements that spans V is a basis for V.

Theorem: Let H be a subspace of a finite-dimensional space V. Any linearly independent set in H can be expanded to a basis for H, and $\dim H \leqslant \dim V$.

Definition: A *one-to-one correspondence* is said to be established between linear spaces V and W if every element $\mathbf{v} \in V$ is mapped to element $\mathbf{w} \in W$ so that each $\mathbf{w} \in W$ is mapped by only one element $\mathbf{v} \in V$. One of the usual notations is: $\mathbf{v} \leftrightarrow \mathbf{w}$.

Definition: Linear spaces V and W are called *isomorphic* if there is a one-to-one correspondence such that:

$(\mathbf{v}_1 + \mathbf{v}_2) \leftrightarrow (\mathbf{w}_1 + \mathbf{w}_2)$ if $\mathbf{v}_i \leftrightarrow \mathbf{w}_i$

and $c\,\mathbf{v} \leftrightarrow c\,\mathbf{w}$ if $\mathbf{v} \leftrightarrow \mathbf{w}$ (with the same c)

Theorem: Linear spaces V and W are isomorphic if and only if $\dim V = \dim W$.

Establishing isomorphism to an appropriate vector space is highly useful in the analysis of linear spaces, since vector spaces can be analysed with the necessary matrix equations.

So, an indexed set $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_p\}$ is a **basis** in V if:
$\mathcal{B}$ is a linearly independent set, and $V = \mathrm{Span}\{\mathbf{b}_1, \ldots \mathbf{b}_p\}$

An important reason for specifying a basis $\mathcal{B}$ for linear space V is to introduce a coordinate system within V:

If $\dim V = n$, then a coordinate system will describe a specific isomorphism between linear space V and vector space $\mathbb{R}^n$.

Theorem: (the *unique representation theorem*)

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ be a basis for a linear space V.

Then $\forall\, \mathbf{x} \in V$ there exists a *unique* set of scalars $x_1, \ldots x_n$ such that

$$\mathbf{x} = x_1\mathbf{b}_1 + \ldots + x_n\mathbf{b}_n.$$

Proof: As $V = \operatorname{Span} \mathcal{B}$ there exists a set of scalars $\{c_i\}^n$ such that

$$\mathbf{x} = c_1 \mathbf{b}_1 + \ldots + c_n \mathbf{b}_n.$$

Suppose another set $\{d_i\}^n$ also satisfies $\mathbf{x} = d_1 \mathbf{b}_1 + \ldots + d_n \mathbf{b}_n$.

Then we can write

$$\begin{aligned} \mathbf{0} &= \mathbf{x} + (-\mathbf{x}) = \mathbf{x} + (-1) \cdot \mathbf{x} \\ &= c_1 \mathbf{b}_1 + \ldots + c_n \mathbf{b}_n - d_1 \mathbf{b}_1 - \ldots - d_n \mathbf{b}_n \\ &= (c_1 - d_1) \mathbf{b}_1 + \ldots + (c_n - d_n) \mathbf{b}_n. \end{aligned}$$

However, because $\mathcal{B}$ is a linearly independent set, all the coefficients $(c_i - d_i)$ must be equal to zero:

$$c_i = d_i \qquad 1 \leqslant i \leqslant n.$$

Therefore, representation $\mathbf{x} = c_1 \mathbf{b}_1 + \ldots + c_n \mathbf{b}_n$ is unique.

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ be a basis for V, and $\mathbf{x} \in V$.
(1) *Coordinates* of $\mathbf{x}$ *relative to basis* $\mathcal{B}$ (or $\mathcal{B}$-coordinates of $\mathbf{x}$) are the coefficients $x_1, \ldots x_n$ such that

$$\mathbf{x} = x_1\mathbf{b}_1 + \ldots + x_n\mathbf{b}_n$$

(2) If $x_1, \ldots x_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then

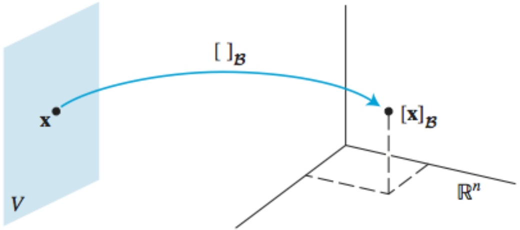
$$[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

is the *coordinate vector* of $\mathbf{x}$, relative to $\mathcal{B}$.

Unique correspondence $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is called *coordinate mapping*.

So, by selecting a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ in a linear space V we can introduce a coordinate system in V.

Coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ makes V isomorphic to $\mathbb{R}^n$.



Linear operations with coordinate vectors of the elements of V are equivalent to the corresponding linear operations with those elements.

Example 1: Coordinates of $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ in the *standard basis* $\mathcal{E}^{(3)}$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are obvious, but can be formally retrieved by "solving" the system

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 = 4 \\ x_2 = 5 \\ x_3 = 6 \end{cases} \quad \text{so} \quad [\mathbf{x}]_{\mathcal{E}} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

Coordinate vector in the standard basis is the same as the vector.

Example 2:

$\mathcal{P} = \{1,\, t,\, t^2,\, t^3,\, t^4\}$ is the standard basis in the space $\mathbb{P}_4$.

A typical element $\mathbf{p}$ of $\mathbb{P}_4$ has the form

$$\mathbf{p} = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$$

This is a linear combination of the standard basis vectors, so

$$[\mathbf{p}]_{\mathcal{P}} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

Coordinate mapping $\mathbf{p} \mapsto [\mathbf{p}]_{\mathcal{P}}$ is an isomorphism of $\mathbb{P}_4$ and $\mathbb{R}^5$.

All linear operations in $\mathbb{P}_4$ correspond to operations in $\mathbb{R}^5$.

Example 3a: Consider a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$, where

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(we can easily see these two vectors are linearly independent)

Let $\mathbf{x} \in \mathbb{R}^2$ have the following $\mathcal{B}$-coordinate vector:

$$[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

The $\mathcal{B}$-coordinates of $\mathbf{x}$ directly produce $\mathbf{x}$ from the vectors of $\mathcal{B}$:

$$\mathbf{x} = -2\mathbf{b}_1 + 3\mathbf{b}_2 = -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

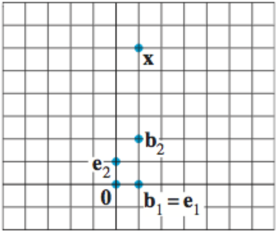
Coordinate mapping: examples

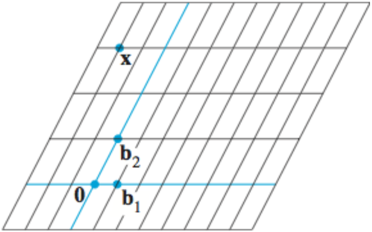
(Example 3a): $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ in the standard basis $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

and in basis $\mathcal{B} = \{\mathbf{b}_1; \mathbf{b}_2\}$ with $\mathbf{b}_1 = \mathbf{e}_1$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Coordinates $\begin{bmatrix} 1 \\ 6 \end{bmatrix}$ locate $\mathbf{x}$ relative to the standard basis.

$\mathcal{B}$-coordinates $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ locate $\mathbf{x}$ relative to $\mathcal{B}$.





Example 3b: The same basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and $\mathbf{x} \in \mathbb{R}^2$:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \qquad \mathbf{x} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

To find the $\mathcal{B}$-coordinates for vector $\mathbf{x}$, we need to solve

$$\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 6 \\ 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} x_1 = 4 \\ x_2 = 2 \end{cases} \quad \text{thus} \quad [\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

This can be easily verified: $\quad 4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

Example 4: recall $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $H = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$

$$\text{with} \quad \mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \quad \text{consider} \quad \mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

and determine if $\mathbf{x} \in H$ and, if so, find $[\mathbf{x}]_{\mathcal{B}}$.

Solution: If $\mathbf{x} \in H$, then the following equation is consistent:

$$c_1 \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$$

The scalars c_1 and c_2, if they exist, are the $\mathcal{B}$ coordinates of $\mathbf{x}$.

$$\left[\begin{array}{rr|r} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \to \left[\begin{array}{rr|r} 3 & -1 & 3 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

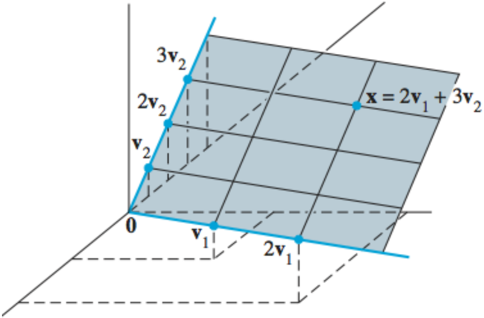
Thus the system is consistent; a unique solution is $c_1 = 2,\ c_2 = 3$.

The coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$ is $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix};$$

and

$$\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix} = 2\mathbf{v}_1 + 3\mathbf{v}_2$$



This illustrates an isomorphism between $\mathrm{Span}\{\mathbf{v}_1,\ \mathbf{v}_2\}$ and $\mathbb{R}^2$.

Example 5:

The first five Chebyshev polynomials (of the first kind) are

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_2(x) = 2x^2 - 1,$$

$$T_3(x) = 4x^3 - 3x, \qquad T_4(x) = 8x^4 - 8x^2 + 1$$

These are polynomials of a degree up to 4, so we can obtain their coordinates in the standard basis of $\mathbb{P}_4$ by expressing each $T_i(x)$ as a linear combination of the elements from $\mathcal{P} = \{1,\, x,\, x^2,\, x^3,\, x^4\}$:

$$[T_i]_{\mathcal{P}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -3 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -8 \\ 0 \\ 8 \end{pmatrix}.$$

Note that we can see these polynomials are linearly independent.

Given different coordinate systems, we can change between them.

The mechanism for this change should be the same for all vectors.

Example: $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}.$

To find the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = (c_1; c_2)$ of $\mathbf{x}$ with respect to basis $\mathcal{B}$, we need to solve the equation $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \mathbf{x}$:

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

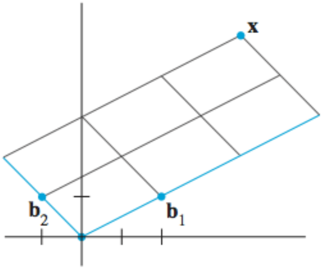
Upon row-reduction, the unique solution is $c_1 = 3$ and $c_2 = 2$.

$$\cdots \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$ ...solution is $c_1 = 3$ and $c_2 = 2$

Thus $\mathbf{x} = 3\mathbf{b}_1 + 2\mathbf{b}_2$

and we can write

$$[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$



Observation:

Multiplication by matrix $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = [\mathbf{b}_1 \quad \mathbf{b}_2]$

converts $\mathcal{B}$-coordinates of any $\mathbf{x}$ into its standard coordinates.

Generally, let $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ be a basis in $\mathbb{R}^n$.

Construct the following matrix: $\mathrm{P}_{\mathcal{B}} = [\mathbf{b}_1, \ldots \mathbf{b}_n]$.

Then the vector equation $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ is equivalent to

$$\mathbf{x} = \mathrm{P}_{\mathcal{B}} \, [\mathbf{x}]_{\mathcal{B}} \qquad \text{where} \qquad [\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$\mathrm{P}_{\mathcal{B}}$ is the *change of coordinates matrix* from $\mathcal{B}$
to the standard basis in $\mathbb{R}^n$: $\{\mathbf{e}_1, \ldots \mathbf{e}_n\}$.

The columns of $\mathrm{P}_{\mathcal{B}}$ are linearly independent (being basis vectors).
Then matrix $\mathrm{P}_{\mathcal{B}}$ is invertible, permitting back-transformation:

Left-multiplication by $\mathrm{P}_{\mathcal{B}}^{-1}$ converts $\mathbf{x}$ into $\mathcal{B}$-coordinate vector

$$[\mathbf{x}]_{\mathcal{B}} = \mathrm{P}_{\mathcal{B}}^{-1}\mathbf{x}$$

Given basis $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ and basis $\mathcal{C} = \{\mathbf{c}_1, \ldots \mathbf{c}_n\}$ for V:

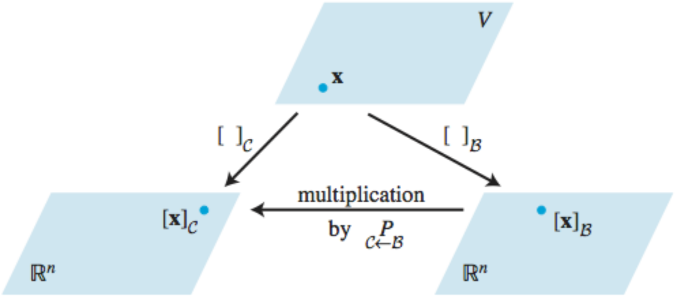
there is a unique $n \times n$ matrix $\mathbf{P}_{\mathcal{C} \leftarrow \mathcal{B}}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C} \leftarrow \mathcal{B}} \, [\mathbf{x}]_{\mathcal{B}}$$

Columns of $\mathbf{P}_{\mathcal{C} \leftarrow \mathcal{B}}$ are the $\mathcal{C}$-coordinates of the vectors of basis $\mathcal{B}$.

That is, $\mathbf{P}_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \ldots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$.

$\mathbf{P}_{\mathcal{C} \leftarrow \mathcal{B}}$ is called the *change of coordinate matrix* from $\mathcal{B}$ to $\mathcal{C}$.



The columns of the square matrix $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ are linearly independent (coordinate vectors of basis $\mathcal{B}$). Therefore $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible.

By multiplying both the sides of $\qquad [\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}$

from the left by $(\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$ we get: $\qquad (\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}\,[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{B}}$

Matrix $(\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}$ converts $\mathcal{C}$-coordinates into $\mathcal{B}$-coordinates:

$$(\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}})^{-1} = \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}$$

Note: For the standard basis $\mathcal{E} = \{\mathbf{e}_1, \ldots \mathbf{e}_n\}$ each $[\mathbf{b}_i]_{\mathcal{E}} = \mathbf{b}_i$.

Thus $\qquad \mathbf{P}_{\mathcal{E}\leftarrow\mathcal{B}} = \Big[[\mathbf{b}_1]_{\mathcal{E}}, \ldots [\mathbf{b}_n]_{\mathcal{E}}\Big] = \big[\mathbf{b}_1, \ldots \mathbf{b}_n\big] = \mathbf{P}_{\mathcal{B}}$

Example: Find $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ from $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ to $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, where

$$\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

Solution: The columns of $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ are $\mathcal{C}$-coordinates of $\mathbf{b}_1$ and $\mathbf{b}_2$:

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ [\mathbf{b}_2]_{\mathcal{C}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \quad \text{Then} \quad \begin{cases} \mathbf{b}_1 = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 \\ \mathbf{b}_2 = y_1\mathbf{c}_1 + y_2\mathbf{c}_2 \end{cases}$$

which means $\begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{b}_1, \qquad \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \mathbf{b}_2.$

To find x_1, x_2, y_1, y_2 all at once, use doubly augmented matrix:

$$\left[\begin{array}{cc|cc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{array} \right].$$

$$\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}.$$

$$[\mathbf{c}_1\ \mathbf{c}_2 \mid \mathbf{b}_1\ \mathbf{b}_2] = \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 3 & -9 & -5 \\ 0 & 7 & -35 & -21 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

Thus

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \end{pmatrix}, \qquad [\mathbf{b}_2]_{\mathcal{C}} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

so the change of coordinate matrix is

$$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}.$$

$$[\mathbf{c}_1\ \mathbf{c}_2 \mid \mathbf{b}_1\ \mathbf{b}_2] = \left[\begin{array}{rr|rr} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \to \left[\begin{array}{rr|rr} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right]$$

Summary:

Matrix $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ appeared in the right-hand side upon row reduction:

$$\left[\begin{array}{cc|cc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{array}\right] \to \left[\begin{array}{rr|rr} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{I} & \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} \end{array}\right]$$

So, the i-th column of $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ results from row reducing $[\,\mathbf{c}_1\ \mathbf{c}_2 \mid \mathbf{b}_i\,]$.

The same procedure is used for a change of basis generally in $\mathbb{R}^n$:

$$\left[\begin{array}{cccc|cccc} \mathbf{c}_1 & \mathbf{c}_2 & \dots & \mathbf{c}_n & \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \end{array}\right] \to \left[\begin{array}{c|c} \mathbf{I} & \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} \end{array}\right]$$

Another way to change coordinates from $\mathcal{B}$ to $\mathcal{C}$ is to combine the transitions into standard coordinates, realised by $\mathbf{P}_{\mathcal{B}}$ and $\mathbf{P}_{\mathcal{C}}$.

$$\forall\, \mathbf{x} \in \mathbb{R}^n \qquad \mathbf{P}_{\mathcal{B}} \left[\mathbf{x}\right]_{\mathcal{B}} = \mathbf{x} \qquad \mathbf{P}_{\mathcal{C}} \left[\mathbf{x}\right]_{\mathcal{C}} = \mathbf{x}$$

From the latter relation we express

$$\left[\mathbf{x}\right]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}}^{-1}\,\mathbf{x}$$

and subsequently using the other relation we obtain

$$\left[\mathbf{x}\right]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}}^{-1}\,\mathbf{x} = \mathbf{P}_{\mathcal{C}}^{-1}\,\mathbf{P}_{\mathcal{B}}\,\left[\mathbf{x}\right]_{\mathcal{B}}$$

Therefore

$$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \mathbf{P}_{\mathcal{C}}^{-1}\,\mathbf{P}_{\mathcal{B}} \qquad \text{and analogously} \qquad \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}} = \mathbf{P}_{\mathcal{B}}^{-1}\,\mathbf{P}_{\mathcal{C}}$$

This approach is however slower than the direct transformation.

Example: Find $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ by using $\mathbf{P}_{\mathcal{C}}$ and $\mathbf{P}_{\mathcal{B}}$ for the two bases:

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \ \mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}; \quad \mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}, \ \mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}.$$

We first construct $\mathbf{P}_{\mathcal{C}}$ and $\mathbf{P}_{\mathcal{B}}$ and then use $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = (\mathbf{P}_{\mathcal{C}})^{-1}\mathbf{P}_{\mathcal{B}}$.

$$\mathbf{P}_{\mathcal{B}} = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}, \qquad \mathbf{P}_{\mathcal{C}} = \begin{bmatrix} -7 & -5 \\ 9 & 7 \end{bmatrix};$$

then

$$\mathbf{P}_{\mathcal{C}}^{-1} = \begin{bmatrix} -7/4 & -5/4 \\ 9/4 & 7/4 \end{bmatrix}$$

and so

$$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = (\mathbf{P}_{\mathcal{C}})^{-1}\,\mathbf{P}_{\mathcal{B}} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix}.$$

Example: Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ such that $\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2$ and $\mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$, and some $\mathbf{x}$, for which we only know that $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$, that is,

$$[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Note that we do not know vectors $\mathbf{b}_i$, $\mathbf{c}_i$, and $\mathbf{x}$ themselves.

Let us apply $\mathcal{C}$ coordinates to $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$. Coordinate mapping introduces isomorphism, which preserves linear operations, so

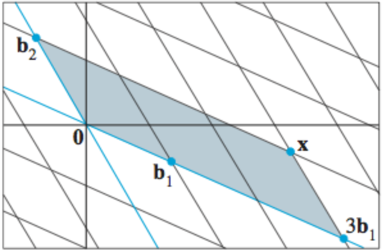
$$[\mathbf{x}]_{\mathcal{C}} = [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} = 3\,[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

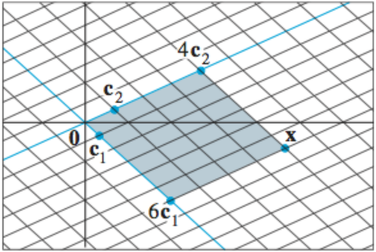
However we know that $\quad [\mathbf{b}_1]_{\mathcal{C}} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{pmatrix} -6 \\ 1 \end{pmatrix}$

Given those $[\mathbf{b}_1]_{\mathcal{C}}$ and $[\mathbf{b}_2]_{\mathcal{C}}$, we have $\begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}$

Thus the coordinates are connected by the above matrix, and

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$





On the left, $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$. On the right, the same $\mathbf{x} = 6\mathbf{c}_1 + 4\mathbf{c}_2$.

- Coordinates

  (coefficients from linear combination of basis elements)

- Change of coordinates

  (achieved by matrix multiplication)

Important: next week is unusual

1. No on-campus lecture on 28 March

   Recordings on Canvas to study online (moved from 25 April):

a) Elementary matrices and matrix representation of row reduction

b) LU decomposition for solving large-scale systems

2. No tutorials on 29 March (public holiday)

   Exercises for self-study of the above topics

3. Upcoming quick tests

   on 22 March: basis, dimensions and isomorphism

   on 5 April: coordinates and change of basis

4. Next lecture on campus: 4 April