University of Technology Sydney

School of Mathematical and Physical Sciences

37233 Linear Algebra

Solutions 5

Question 1

(a) Forming the corresponding matrix and proceeding with row reduction we see pivots

$$\begin{bmatrix} 1 & -3 & 2 \\ -1 & 4 & -2 \\ -3 & 9 & 4 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

in every column, so the set is linearly independent, spans $\mathbb{R}^3$, and thus is a basis for $\mathbb{R}^3$.

(b) For this we solve the corresponding equation, row-reducing the augmented matrix

$$\begin{bmatrix} 1 & -3 & 2 & 8 \\ -1 & 4 & -2 & -9 \\ -3 & 9 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 & 8 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 10 & 30 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix} \quad \text{so} \quad [\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$$

(c) For this we proceed with direct multiplication

$$\begin{bmatrix} 1 & -3 & 2 \\ -1 & 4 & -2 \\ -3 & 9 & 4 \end{bmatrix} \begin{pmatrix} 16 \\ 5 \\ 1 \end{pmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \qquad \text{so} \quad \mathbf{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

Question 2

(a) $$[\mathbf{x}]_{\mathcal{C}} = -6\,[\mathbf{b}_1]_{\mathcal{C}} + 5\,[\mathbf{b}_2]_{\mathcal{C}} - [\mathbf{b}_3]_{\mathcal{C}} = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -3 & -4 \\ 2 & 4 & 7 \end{bmatrix} \begin{pmatrix} -6 \\ 5 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Equivalently, this can be verified by direct calculation as:
$\mathbf{x} = -6(\mathbf{c}_1 - 2\mathbf{c}_2 + 2\mathbf{c}_3) + 5(2\mathbf{c}_1 - 3\mathbf{c}_2 + 4\mathbf{c}_3) - (3\mathbf{c}_1 - 4\mathbf{c}_2 + 7\mathbf{c}_3) = \mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3\,.$

(b) Note the matrix appeared in (b) is the change of coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. The change of coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$ is the inverse of it, which we can obtain by row-reducing $\left[\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}\,|\,\mathbf{I}\right] \rightarrow \left[\mathbf{I}\,|\,\mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}\right]$:

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ -2 & -3 & -4 & 0 & 1 & 0 \\ 2 & 4 & 7 & 0 & 0 & 1 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & -5 & -2 & 1 \\ 0 & 1 & 0 & 6 & 1 & -2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{bmatrix}$$

Then $[\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} -5 & -2 & 1 \\ 6 & 1 & -2 \\ -2 & 0 & 1 \end{bmatrix} \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$, equivalently $\mathbf{y} = 3\mathbf{b}_1 - \mathbf{b}_2\,.$

Question 3

(a) Row-reduction of these matrices shows pivots in every column

$$\mathbf{B} \to \begin{bmatrix} 3 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4/3 \\ 0 & 0 & 2 & -16/3 \\ 0 & 0 & 0 & -1/3 \end{bmatrix} \to \mathbf{I} \quad \text{and} \quad \mathbf{C} \to \begin{bmatrix} 1 & 5 & -5 & 5 \\ 0 & -2 & 4 & -4 \\ 0 & 0 & -44 & 48 \\ 0 & 0 & 0 & -2/11 \end{bmatrix} \to \mathbf{I}$$

so both the columns of $\mathbf{B}$ and the columns of $\mathbf{C}$ form bases for $\mathbb{R}^4$.

(b) From solving $\mathbf{B}\,[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$ we find $[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 5 \\ 1 \end{pmatrix}$.

(c) $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ is found by row-reducing

$$[\,\mathbf{C}\,|\,\mathbf{B}\,] \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(d) $[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 2 \\ 7 \\ 6 \end{pmatrix}$.

Question 4

(a) Write the coordinates of Hermite polynomials in the standard polynomial basis, and form the corresponding matrix:

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

There are pivots in every column, so this is a linearly independent set. This set spans the space $\mathbb{P}^3$ because there are pivots in every row. Thus the set is a basis for $\mathbb{P}^3$.

(b) The coordinates of $p(t)$ in the standard basis are $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

To find the $\mathcal{H}$-coordinates of $p(t)$ we need to solve the system:

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 2 & 0 & -12 & 1 \\ 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 8 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 3/2 \\ 0 & 1 & 0 & 0 & 5/4 \\ 0 & 0 & 1 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 1/8 \end{bmatrix} \quad \Rightarrow \quad [p]_{\mathcal{H}} = \begin{pmatrix} 3/2 \\ 5/4 \\ 1/4 \\ 1/8 \end{pmatrix}$$

which can be easily verified by calculating the corresponding linear combination.

(c) To find q from its $\mathcal{H}$-coordinates we take a direct multiplication by the matrix $\mathbf{H}$:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{bmatrix} -1 \\ -10 \\ 4 \\ 8 \end{bmatrix}$$

which is the same as summing up the Hermite polynomials: $q(t) = 8t^3 + 4t^2 - 10t - 1$.

(d)

(i) Each of the q_i cannot be obtained as a linear combination of the previous vectors (as there is a term with higher power added each time), so the set is linearly independent. These vectors span the space of polynomials $\mathbb{P}^3$ so they form a basis. Coordinates of this set it terms of the standard polynomial basis, written together as columns of a matrix, are

$$\mathbf{Q} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Coordinates of the set of Hermite polynomials it terms of the standard polynomial basis, written together as columns of a matrix, are

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

so the change of coordinates matrix from basis $\mathcal{H}$ to basis $\mathcal{Q}$ is then found from row reduction $\left[\,\mathbf{Q}\,|\,\mathbf{H}\,\right] \to \left[\,\mathbf{I}\,|\,\mathbf{P}_{\mathcal{Q}\leftarrow\mathcal{H}}\,\right]$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 1 & 0 & 2 & 0 & -12 \\ 0 & 0 & 1 & 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 8 \end{array}\right] \to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -2 & -2 & 12 \\ 0 & 1 & 0 & 0 & 0 & 2 & -4 & -12 \\ 0 & 0 & 1 & 0 & 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 8 \end{array}\right]$$

(iii) Coordinates of r in the standard polynomial basis are $(0; 1; 2; 3)$ so its $\mathcal{H}$-coordinates are found by solving

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix} [r]_{\mathcal{H}} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \qquad \Rightarrow \qquad [r]_{\mathcal{H}} = \begin{pmatrix} 1 \\ 11/4 \\ 1/2 \\ 3/8 \end{pmatrix}$$

(iv)

$$[r]_{\mathcal{Q}} = \mathbf{P}_{\mathcal{Q}\leftarrow\mathcal{H}}\,[r]_{\mathcal{H}} = \begin{bmatrix} 1 & -2 & -2 & 12 \\ 0 & 2 & -4 & -12 \\ 0 & 0 & 4 & -8 \\ 0 & 0 & 0 & 8 \end{bmatrix} \begin{pmatrix} 1 \\ 11/4 \\ 1/2 \\ 3/8 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 3 \end{pmatrix}$$