University of Technology Sydney
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37233 Linear Algebra

Tutorial 5 — solution guide

Question 1

(a) This is done by solving the system

$$\begin{bmatrix} -2 & 7 \\ 1 & -2 \end{bmatrix} [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \Rightarrow \qquad [\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

(b) $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ is obtained from row-reducing $\left[\,\mathbf{c}_1\,\mathbf{c}_2 \,|\, \mathbf{b}_1\,\mathbf{b}_2\,\right]$:

$$\begin{bmatrix} -1 & 4 & -2 & 7 \\ -1 & 1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & -1 & 3 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix}$$

Then

$$[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 & 5 \\ -1 & 3 \end{bmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

(c) This is done by solving the system

$$\begin{bmatrix} -1 & 4 \\ -1 & 1 \end{bmatrix} [\mathbf{y}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \qquad \Rightarrow \qquad [\mathbf{y}]_{\mathcal{C}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

(d) $\mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}$ is obtained from row-reducing $\left[\,\mathbf{b}_1\,\mathbf{b}_2 \,|\, \mathbf{c}_1\,\mathbf{c}_2\,\right]$:

$$\begin{bmatrix} -2 & 7 & -1 & 4 \\ 1 & -2 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 5 \\ 0 & 1 & -1 & 2 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix}$$

Then

$$[\mathbf{y}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}\,[\mathbf{y}]_{\mathcal{C}} = \begin{bmatrix} -3 & 5 \\ -1 & 2 \end{bmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Question 2

Basis vectors are linearly independent, so $\mathbf{b}_i = \mathbf{b}_i + \sum_{j\neq i} 0 \cdot \mathbf{b}_j$ which tells that the only non-zero coordinate of $\mathbf{b}_i$ in $\mathcal{B}$ is the i-th component, equal to 1.

Alternatively, $[\mathbf{b}_i]_{\mathcal{B}} = \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{E}}\mathbf{b}_i$ so then $\mathbf{b}_i = \mathbf{P}_{\mathcal{E}\leftarrow\mathcal{B}}\,[\mathbf{b}_i]_{\mathcal{B}} = \mathbf{P}_{\mathcal{B}}\,[\mathbf{b}_i]_{\mathcal{B}} = \left[\mathbf{b}_1 \ldots \mathbf{b}_n\right] [\mathbf{b}_i]_{\mathcal{B}}$ which is only true when $[\mathbf{b}_i]_{\mathcal{B}} = \mathbf{e}_i$.

Question 3

(a) The change-of-coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$ is composed by $\mathcal{C}$-coordinates of the vectors from $\mathcal{B}$. These are directly known from the specified linear relations:

$$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \left[\, [\mathbf{b}_1]_{\mathcal{C}} \quad [\mathbf{b}_2]_{\mathcal{C}} \,\right] = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix}$$

The change-of-coordinate matrix from $\mathcal{C}$ to $\mathcal{B}$ is constructed as in inverse of $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$, equivalently by taking the right part after row-reducing $[\,\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}\,|\,\mathbf{I}\,] \to [\,\mathbf{I}\,|\,\mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}\,]$:

$$\begin{bmatrix} -1 & 2 & 1 & 0 \\ 3 & -4 & 0 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3/2 & 1/2 \end{bmatrix}$$

(b) $[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Equivalently, $\mathbf{x} = 5(-\mathbf{c}_1 + 3\mathbf{c}_2) + 3(2\mathbf{c}_1 - 4\mathbf{c}_2) = \mathbf{c}_1 + 3\mathbf{c}_2\,.$

(c) $[\mathbf{y}]_{\mathcal{B}} = \mathbf{P}_{\mathcal{B}\leftarrow\mathcal{C}}\,[\mathbf{y}]_{\mathcal{C}} = \begin{bmatrix} 2 & 1 \\ 3/2 & 1/2 \end{bmatrix} \begin{pmatrix} 3 \\ -5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$

Question 4

In terms of a standard basis $\mathcal{M}$ of the linear space $\mathbb{M}_2^2$, such as e.g.

$$\mathbf{M}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{M}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{M}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{M}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

the elements of set $\{\mathbf{W}_i\}$ are represented by the coordinate vectors:

$$[\mathbf{W}_1]_{\mathcal{M}} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \qquad [\mathbf{W}_2]_{\mathcal{M}} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \qquad [\mathbf{W}_3]_{\mathcal{M}} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \qquad [\mathbf{W}_4]_{\mathcal{M}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

so then finding the coordinates of $\mathbf{Y}$ in this basis is achieved by row-reducing

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 0 & 6 \\ 3 & 3 & 0 & 0 & 6 \\ 4 & 0 & 0 & 0 & 4 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \Rightarrow \qquad [\mathbf{Y}]_{\mathcal{W}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Question 5

(a) Writing the coordinates of Chebyshev polynomials relative to the standard basis:

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 2 & 0 & -8 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

we see that the matrix has pivots in every column, so this set is a basis for $\mathbb{P}_4$.

(b) The required matrix is the inverse of the above, found by row-reducing $[\,\mathbf{T}\,|\,\mathbf{I}\,]$:

$$\mathbf{P}_{\Theta\leftarrow\mathcal{P}} \begin{bmatrix} 1 & 0 & 1/2 & 0 & 3/8 \\ 0 & 1 & 0 & 3/4 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1/8 \end{bmatrix}$$

(c) Given that $\cos\alpha \approx 1 - \dfrac{\alpha^2}{2!} + \dfrac{\alpha^4}{4!}$, we write $[f]_{\mathcal{P}} = \begin{pmatrix} 1 \\ 0 \\ -8 \\ 0 \\ 32/3 \end{pmatrix}$.

(d) Then $[f]_\Theta = \mathbf{P}_{\Theta\leftarrow\mathcal{P}}\,[f]_{\mathcal{P}} = \begin{pmatrix} 1 \\ 0 \\ 4/3 \\ 0 \\ 4/3 \end{pmatrix}$.