University of Technology Sydney
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37233 Linear Algebra

Self-study problems — solution guide

Question 1

(a) For the row reduction, let us first eliminate $a_{21} = 2$ and then $a_{31} = 3$ with the help of row 1, which requires multiplication by

$$\mathbf{E}_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ 0 & -7 & -5 \\ 0 & -6 & -4 \end{bmatrix}$$

and then we need to eliminate the $\{3,\, 2\}$ element using row 2, which is done by

$$\mathbf{E}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6/7 & 1 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{U} = \mathbf{E}_3\mathbf{E}_2\mathbf{E}_1\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 \\ 0 & -7 & -5 \\ 0 & 0 & 2/7 \end{bmatrix}$$

(b) $\mathbf{L}$ is constructed from $\mathbf{E}_i$ matrices; it is easier to calculate $\mathbf{L}$ directly from the inverse $\mathbf{E}_i^{-1}$ matrices, rather than to get $\mathbf{L}^{-1}$ from the $\mathbf{E}_i$ and then inverting $\mathbf{L}^{-1}$. The inverse $\mathbf{E}_i^{-1}$ matrices are straightforward:

$$\mathbf{E}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{E}_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{E}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6/7 & 1 \end{bmatrix}$$

and then

$$\mathbf{L} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}\mathbf{E}_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 6/7 & 1 \end{bmatrix}$$

and we can now check that $\mathbf{LU} = \mathbf{A}$ indeed.

(see next page)

Question 2

(a) Taking the Doolittle approach should result in

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{U} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

which makes it possible to solve $\mathbf{Ly} = \mathbf{b}$ equation, and then $\mathbf{Ux} = \mathbf{y}$ equation, obtaining

$$\mathbf{y} = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \qquad \text{and then} \qquad \mathbf{x} = \begin{bmatrix} 11/3 \\ -7/3 \\ -2 \end{bmatrix}$$

(b) Taking the Crout approach should result in

$$\mathbf{L} = \begin{bmatrix} 2 & 0 & 0 \\ -6 & -3 & 0 \\ 10 & 6 & -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{U} = \begin{bmatrix} 1 & 1/2 & 3/2 \\ 0 & 1 & -4/3 \\ 0 & 0 & 1 \end{bmatrix}$$

which makes it possible to solve $\mathbf{Lz} = \mathbf{b}$ equation, and then $\mathbf{Ux} = \mathbf{z}$ equation, obtaining

$$\mathbf{z} = \begin{bmatrix} -1/2 \\ 1/3 \\ -2 \end{bmatrix} \qquad \text{and then} \qquad \mathbf{x} = \begin{bmatrix} 11/3 \\ -7/3 \\ -2 \end{bmatrix}$$

which, of course, is the same final solution, while $\mathbf{y} \neq \mathbf{z}$.

Question 3

Following the Cholesky algorithm, the matrix for decomposition is $\mathbf{A} = \mathbf{U}^\mathsf{T}\mathbf{U}$ with

$$\mathbf{U} = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{bmatrix}$$

which can be easily verified and subsequently used to solve any $\mathbf{Ax} = \mathbf{b}$ equations.