# Linear transformations

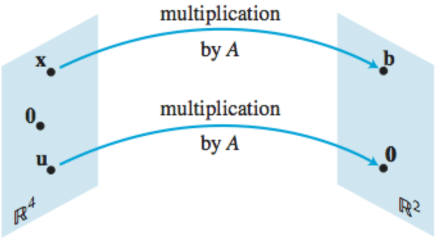
- Linear transformations
- Eigenvectors and eigenvalues

- Equation $\mathbf{Ax} = \mathbf{b}$ can be regarded as a function:

  Matrix $\mathbf{A}$ *acts* on $\mathbf{x}$, producing a new vector $\mathbf{b}$
  (analogous to an action of a function $y = f(x)$)

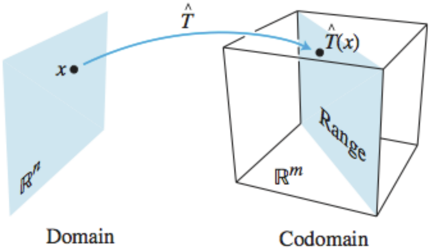
- So, left-multiplication by $\mathbf{A}$ *transforms* $\mathbf{x}$ into $\mathbf{b}$

  The resulting correspondence between $\mathbf{x}$ and $\mathbf{b}$
  is a *mapping* from one set of vectors to another



**Definition:** Transformation (mapping, function) $\widehat{T}$ from $V$ onto $W$ is a rule assigning element $\widehat{T}(\mathbf{x}) = \mathbf{y} \in W$ to each element $\mathbf{x} \in V$

- The usual notation is: $\widehat{T} : V \mapsto W$
- $V$ is called the **domain**, and $W$ the **codomain** of $\widehat{T}$
- Vector $\mathbf{y} = \widehat{T}(\mathbf{x})$ is called the **image** of $\mathbf{x}$
- The set of all images $\left\{\widehat{T}(\mathbf{x})\right\}$ is called the **range** of $\widehat{T}$




Picture:

$V = \mathbb{R}^n$

$W = \mathbb{R}^m$

**Definition:**

Transformation $\widehat{T}$ is **linear** if: $\forall\, \mathbf{u},\ \mathbf{v}$ in the domain of $\widehat{T}$

(i) $\widehat{T}(\mathbf{u}+\mathbf{v}) = \widehat{T}(\mathbf{u}) + \widehat{T}(\mathbf{v})$

(ii) $\widehat{T}(c\,\mathbf{u}) = c\,\widehat{T}(\mathbf{u}) \quad \forall\, c \in \mathbb{R}$

Properties:

If $\widehat{T}$ is a linear transformation, then

- $\widehat{T}(\mathbf{0}) = \mathbf{0}$ ( proof: $\widehat{T}(\mathbf{0}) = \widehat{T}(0 \cdot \mathbf{0}) = 0 \cdot \widehat{T}(\mathbf{0}) = \mathbf{0}$ )
- $\widehat{T}(c\,\mathbf{u} + d\,\mathbf{v}) = c\,\widehat{T}(\mathbf{u}) + d\,\widehat{T}(\mathbf{v})$ ( follows from (i) and (ii) )

Note: Generally, a linear transformation is often called a linear operator

If $\mathbf{A}$ is an $m \times n$ matrix, $\widehat{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$ is a linear transformation.

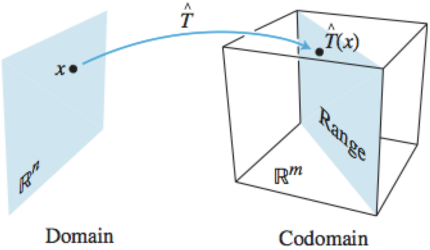
Proof: given that $\mathbf{A}\mathbf{x} = x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n$

$$\begin{aligned}\widehat{T}(\mathbf{x}+\mathbf{y}) &= (x_1+y_1)\mathbf{a}_1 + \ldots + (x_n+y_n)\mathbf{a}_n \\ &= x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n + y_1\mathbf{a}_1 + \ldots + y_n\mathbf{a}_n = \widehat{T}(\mathbf{x}) + \widehat{T}(\mathbf{y})\end{aligned}$$

$$\begin{aligned}\widehat{T}(c\,\mathbf{x}) &= (cx_1)\mathbf{a}_1 + \ldots + (cx_n)\mathbf{a}_n \\ &= c\,(x_1\mathbf{a}_1 + \ldots + x_n\mathbf{a}_n) = c\,T(\mathbf{x})\end{aligned}$$

Thus $\widehat{T}(\mathbf{x}+\mathbf{y}) = \widehat{T}(\mathbf{x}) + \widehat{T}(\mathbf{y})$ and $\widehat{T}(c\,\mathbf{x}) = c \cdot \widehat{T}(\mathbf{x})$
so $\widehat{T} = \mathbf{A}\mathbf{x}$ is a linear transformation, by definition.

- For $\mathbf{x} \in \mathbb{R}^n$, a linear transformation $\widehat{T}(\mathbf{x})$ into $\mathbb{R}^m$ can be computed as $\mathbf{Ax}$ where $\mathbf{A}$ is an $m \times n$ matrix.
- If the domain of $\widehat{T}$ is $\mathbb{R}^n$ then $\mathbf{A}$ has $n$ columns, and if the codomain of $\widehat{T}$ is $\mathbb{R}^m$ then $\mathbf{A}$ has $m$ rows.
- The range of $\widehat{T}$ is $\mathrm{Span}\{\mathbf{a}_1 \ldots \mathbf{a}_n\}$ and image of $\mathbf{x}$ is $\mathbf{Ax}$.



**Example 1:** image of $\mathbf{u}$ obtained with $\mathbf{A}$:

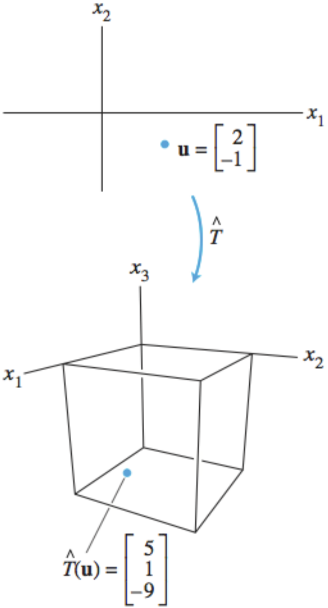
$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

This is a $\widehat{T} : \mathbb{R}^2 \mapsto \mathbb{R}^3$ transformation:

$$\widehat{T}(\mathbf{u}) = \mathbf{A}\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

This can be also written in functional form:

$$\widehat{T}(\mathbf{x}) = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$



**Example 2:** Let $\widehat{T}(x_1, x_2) = \big((2x_1 - 3x_2);\ (x_1 + 4);\ (5x_2)\big)$.

Show that the above $\widehat{T}$ is *not a linear* transformation:

$$\widehat{T}(x_1, x_2) = \begin{bmatrix} 2x_1 - 3x_2 \\ x_1 + 4 \\ 5x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

Thus $\widehat{T}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{q}$ where

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{q} = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix}$$

If $\widehat{T}$ were a linear transformation, then $\widehat{T}(\mathbf{0}) = \mathbf{0}$

However, here $\widehat{T}(\mathbf{0}) = \mathbf{q} \neq \mathbf{0}$

Therefore, it is not a linear transformation.

**Example 3:** $\mathbf{T} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}, \quad \widehat{T}(\mathbf{x}) = \mathbf{T}\mathbf{x}.$

To find $\mathbf{x}$ such that $\widehat{T}(\mathbf{x}) = \mathbf{b}$ we need to solve the system:

$$\left[\begin{array}{rr|r} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{rr|r} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{array}\right]$$

So $x_1 = \dfrac{3}{2}$, $x_2 = -\dfrac{1}{2}$ and thus $\mathbf{b}$ is the image of $\mathbf{x} = \begin{bmatrix} 1.5 \\ -0.5 \end{bmatrix}$

A unique solution implies only one $\mathbf{x}$ with the image $\mathbf{b}$.

We have also verified that $\mathbf{b}$ is in the range of $\widehat{T}$.

**Example 4:** $\mathbf{T} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}, \quad \widehat{T}(\mathbf{x}) = \mathbf{T}\mathbf{x}.$

Determine whether or not $\mathbf{c}$ is in the range of transformation $\widehat{T}$.

Vector $\mathbf{c}$ is in the range of $\widehat{T}(\mathbf{x})$ if $\mathbf{c}$ is an image of some $\mathbf{x} \in \mathbb{R}^2$.

Solving the system $\mathbf{T}\mathbf{x} = \mathbf{c}$ gives

$$\left[\begin{array}{rr|r} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{rr|c} 1 & -3 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 10 \end{array}\right]$$

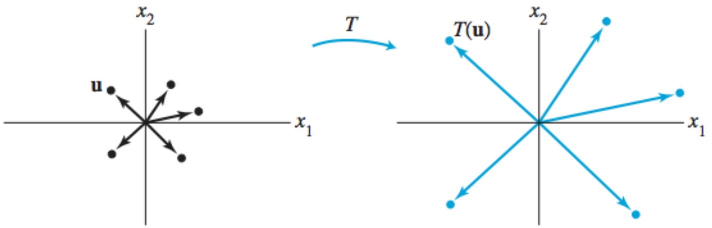
The system is inconsistent and thus $\mathbf{c}$ is **not** in the range of $\widehat{T}$.
There are no vectors in $\mathbb{R}^2$ with an image $\mathbf{c}$ under $\widehat{T}$.

**Example:** Let $\widehat{T} : \mathbb{R}^2 \mapsto \mathbb{R}^2$, $\widehat{T}(\mathbf{x}) = r\,\mathbf{x}$, where $r \in \mathbb{R}$.

Show that this is a linear transformation:

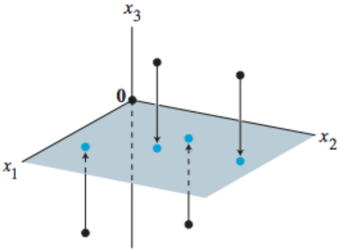
$$\widehat{T}(c\,\mathbf{u} + d\,\mathbf{v}) = r(c\,\mathbf{u} + d\,\mathbf{v}) = c(r\,\mathbf{u}) + d(r\,\mathbf{v}) = c\,\widehat{T}(\mathbf{u}) + d\,\widehat{T}(\mathbf{v}).$$

$\widehat{T}$ is called a *scaling* (or dilation) transformation:



**Example**:

$$\mathbf{O_{12}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$




Action of this matrix on $\mathbf{x} \in \mathbb{R}^3$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

This transformation projects points of $\mathbb{R}^3$ onto a plane.

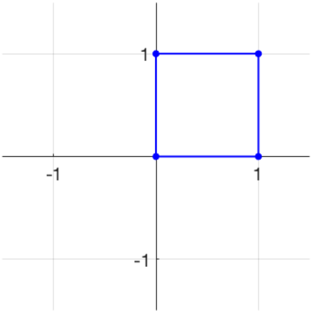
This is an example of *projection* transformation.

Note: this is still a $\mathbb{R}^3$ onto $\mathbb{R}^3$ transformation; its domain is $\mathbb{R}^3$ and codomain is $\mathbb{R}^3$, but the range is a plane within $\mathbb{R}^3$.

A convenient way to visualise transformations in $\mathbb{R}^2$ is to depict their action on a unit square: a square made by vectors

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



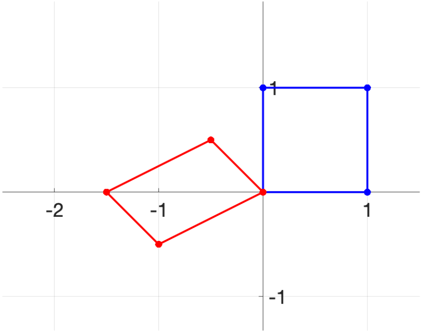

Plotting the resulting points and connecting these with lines (transformation is linear) shows how the shape is changed.

# Visualisation of linear transformations

Example: illustrate linear transformation given by $\begin{bmatrix} -1 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$

$$\begin{bmatrix} -1 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -0.5 \end{bmatrix}$$

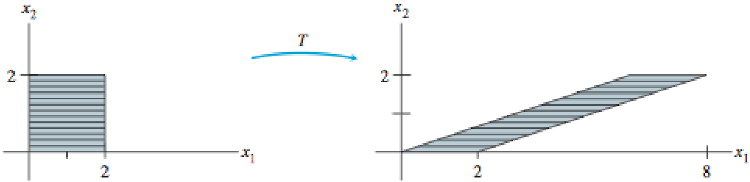



$$\begin{bmatrix} -1 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightrightarrows \begin{bmatrix} -1.5 \\ 0 \end{bmatrix} \leftleftarrows \begin{bmatrix} -1 \\ -0.5 \end{bmatrix} + \begin{bmatrix} -0.5 \\ 0.5 \end{bmatrix}$$

**Example:** transformation with $\mathbf{T} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$ of a $2 \times 2$ square

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$




This is an example of *shear* transformation (square to parallelogram)

# Standard matrices for linear transformations

- So far, we have seen how to visualise a given transformation
- But it is also important to design a desired transformation
- There is a straightforward algorithm: standard matrices

In $\mathbb{R}^2$, any $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$ where

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are called the *standard basis vectors in* $\mathbb{R}^2$.

Similarly, in $\mathbb{R}^3$, any $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ with the basis

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Generally in $\mathbb{R}^n$ $\quad \mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n \quad$ with $\quad \mathbf{e}_n : \begin{cases} e_n^{(i=n)} = 1 \\ e_n^{(i\neq n)} = 0 \end{cases}$

For any linear transformation $\widehat{T}: \mathbb{R}^2 \mapsto \mathbb{R}^2$ we can write

$$\widehat{T}(\mathbf{x}) = \widehat{T}(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = \widehat{T}(x_1\mathbf{e}_1) + \widehat{T}(x_2\mathbf{e}_2) = x_1\widehat{T}(\mathbf{e}_1) + x_2\widehat{T}(\mathbf{e}_2)$$

This can be written as $\widehat{T}(\mathbf{x}) = \left[\begin{array}{c|c} \widehat{T}(\mathbf{e}_1) & \widehat{T}(\mathbf{e}_2) \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right] \equiv \mathbf{T}\mathbf{x}$

$\mathbf{T} = \left[\begin{array}{c|c} \widehat{T}(\mathbf{e}_1) & \widehat{T}(\mathbf{e}_2) \end{array}\right]$ is called the *standard matrix* of $\widehat{T}$.

Similarly, for any linear transformation $\widehat{T}: \mathbb{R}^n \mapsto \mathbb{R}^n$ we can write

$$\begin{aligned} \widehat{T}(\mathbf{x}) = \widehat{T}(x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n) &= \widehat{T}(x_1\mathbf{e}_1) + \ldots + \widehat{T}(x_n\mathbf{e}_n) \\ &= x_1\widehat{T}(\mathbf{e}_1) + \ldots + x_n\widehat{T}(\mathbf{e}_n) = \mathbf{T}\mathbf{x} \end{aligned}$$

with the standard matrix $\mathbf{T} = \left[\begin{array}{c|c|c} \widehat{T}(\mathbf{e}_1) & \ldots & \widehat{T}(\mathbf{e}_n) \end{array}\right]$.

**Example:** Find the standard matrix for scaling $\widehat{T}(\mathbf{x}) = r\mathbf{x}$

Action of this scaling on standard vectors is straightforward:

$$\widehat{T}(\mathbf{e}_1) = r\mathbf{e}_1 = r\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$$

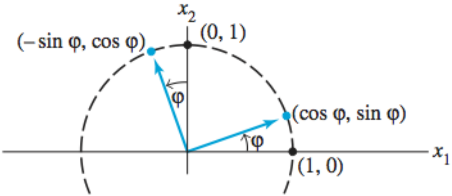
$$\widehat{T}(\mathbf{e}_2) = r\mathbf{e}_2 = r\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

The standard matrix is therefore:

$$\mathbf{D}_r = \left[\widehat{T}(\mathbf{e}_1) \;\middle|\; \widehat{T}(\mathbf{e}_2)\right] = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

Find the standard matrix of a transformation that rotates any vector in $\mathbb{R}^2$ around the origin by an angle $\varphi$ radians.

Consider the action of this rotation over the standard vectors:




$$\widehat{T}(\mathbf{e}_1) = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \quad \text{and} \quad \widehat{T}(\mathbf{e}_2) = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$$

Therefore

$$\mathbf{R}_{\varphi} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

**Example:** counterclockwise rotation by $90^\circ$ about the origin

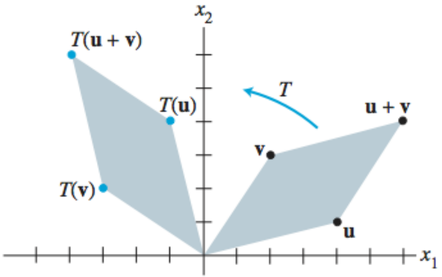
$$\mathbf{T} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{so} \quad \widehat{T}(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

Find the images of: $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}+\mathbf{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$

$$\widehat{T}(\mathbf{u}) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

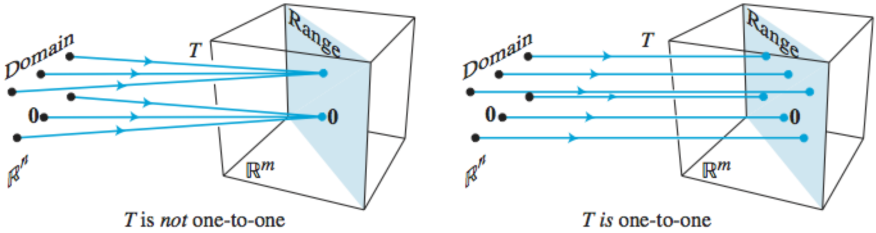
$$\widehat{T}(\mathbf{v}) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\widehat{T}(\mathbf{u}+\mathbf{v}) = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$$



# One-to-one linear transformations

**Definition:** Transformation $\widehat{T} : V \mapsto W$ is called *one-to-one* if each $\mathbf{y} \in W$ is the image of at most one $\mathbf{v} \in V$




**Theorem:** A linear transformation $\widehat{T}$ is one-to-one if and only if the equation $\widehat{T}(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $(\mathbf{x} = \mathbf{0})$.

Equivalent **definition** for vector spaces:

Transformation $\widehat{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a one-to-one linear transformation if $\forall\, \mathbf{b} \in \mathbb{R}^m$ equation $\mathbf{Tx} = \mathbf{b}$ (where $\mathbf{T}$ is the matrix of $\widehat{T}$) has either a unique solution or no solutions.

**Theorem:** Let $\widehat{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation with the standard matrix $\mathbf{T}$. Then:

(a) $\widehat{T}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $\mathbf{T}$ span $\mathbb{R}^m$

(b) $\widehat{T}$ is one-to-one if and only if the columns of $\mathbf{T}$ are linearly independent

- Isomorphism between linear spaces is established with a one-to-one linear transformation
- Coordinate mapping is a one-to-one linear transformation
- Change of basis is a one-to-one linear transformation (and change of basis matrix is the matrix of transformation)

Linearity of transformation ensures that coordinate vector of a linear combination of elements is a linear combinations of their coordinate vectors, taken with the same coefficients:

$$[c_1\mathbf{u}_1 + \ldots + c_n\mathbf{u}_n]_{\mathcal{B}} = c_1 [\mathbf{u}_1]_{\mathcal{B}} + \ldots + c_n [\mathbf{u}_n]_{\mathcal{B}}$$

**Example:** Consider a transformation $\widehat{T}$ with standard matrix

$$\mathbf{T} = \begin{bmatrix} 1 & -4 & 8 & 1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

Check if this transformation is a one-to-one linear transformation, and specify the domain, codomain, and range of $\widehat{T}$.

- $\widehat{T}$ is presented as $\mathbf{Tx}$ so it is a linear transformation.
- The domain of $\widehat{T}$ is $\mathbb{R}^4$ and the codomain is $\mathbb{R}^3$.
- There are three pivots in $\mathbf{T}$ so the columns of $\mathbf{T}$ span $\mathbb{R}^3$, thus the range of $\widehat{T}$ is the entire $\mathbb{R}^3$.
- The columns of $\mathbf{T}$ are not linearly independent, therefore $\widehat{T}$ is **not** one-to-one.

**Example:** Check if the transformation $\widehat{T}(x_1, x_2) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix}$ is a one-to-one linear transformation, and find its range.

This transformation can be written in a matrix form:

$$\widehat{T}(\mathbf{x}) = \begin{bmatrix} 3x_1 + x_2 \\ 5x_1 + 7x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{T}\mathbf{x}$$

- The columns of $\mathbf{T}$ are linearly independent therefore $\widehat{T}$ is a one-to-one linear transformation.
- $\mathbf{x} \in \mathbb{R}^2$ so domain of $\widehat{T}$ is $\mathbb{R}^2$; $\mathbf{T}\mathbf{x} \in \mathbb{R}^3$ so codomain is $\mathbb{R}^3$.
- The two linearly independent columns of $\mathbf{T}$ span a plane within $\mathbb{R}^3$: this plane is the range of $\widehat{T}$.

(this topic officially is a pre-requisite knowledge

— but there are some new details)

**Definition:** An *eigenvector* of a square matrix $\mathbf{A}$ is a nonzero vector $\mathbf{v}$ such that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ for some scalar $\lambda$.

A scalar $\lambda$ is called an *eigenvalue* of $\mathbf{A}$ if there is a nontrivial solution $\mathbf{v}$ of $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.

(then this $\mathbf{v}$ is an eigenvector *corresponding* to $\lambda$).

As a *linear transformation* $\mathbf{A}$ performs *scaling* of its eigenvectors

$\lambda$ is an eigenvalue for $\mathbf{A}$ if and only there is a nontrivial solution to

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

Therefore, $\lambda$ can be found by solving a *characteristic equation*:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Note: there may be complex roots to the characteristic equation.

**Example:** Find eigenvalues and eigenvectors for $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$.

**Solution:** Construct characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$:

$$\left| \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} 2-\lambda & 5 \\ 6 & 1-\lambda \end{bmatrix} \right| = 0$$

$$(2-\lambda)(1-\lambda) - 30 = 0$$

$$\lambda^2 - 3\lambda - 28 = 0$$

which yields the eigenvalues as the solution:

$$\lambda_1 = 7 \qquad \lambda_2 = -4$$

For each of these eigenvalues, we need to solve $(\mathbf{A} - \lambda\mathbf{I})\,\mathbf{x} = \mathbf{0}$.

**Example:** For $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$ we found $\lambda_1 = 7, \quad \lambda_2 = -4$

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$$\mathbf{A} - 7\mathbf{I} = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -5 & 5 \\ 6 & -6 \end{bmatrix}$$

$$\to \quad \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\mathbf{A} + 4\mathbf{I} = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 6 & 5 \end{bmatrix}$$

$$\to \quad \begin{bmatrix} 6 & 5 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v}_2 = c \cdot \begin{bmatrix} 5 \\ -6 \end{bmatrix}$$

**Example:** Find eigenvalues and eigenvectors for $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix}$.

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So the eigenvalues are: $\lambda_1 = 7$ and $\lambda_2 = -4$

and the corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$

We can verify this back with the original matrix:

$$\mathbf{A}\mathbf{v}_1 = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda_1 \mathbf{v}_1$$

$$\mathbf{A}\mathbf{v}_2 = \begin{bmatrix} 2 & 5 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -6 \end{bmatrix} = \begin{bmatrix} -20 \\ 24 \end{bmatrix} = -4 \cdot \begin{bmatrix} 5 \\ -6 \end{bmatrix} = \lambda_2 \mathbf{v}_2$$

# Eigenvectors and eigenvalues



- $0$ is an eigenvalue of $\mathbf{A}$ if and only if $\mathbf{A}$ is singular.

- All the eigenvectors of $\mathbf{A}$ corresponding to a given eigenvalue, span the corresponding *eigenspace*

- If $\mathbf{v}_1, \dots \mathbf{v}_p$ are eigenvectors corresponding to different eigenvalues $\lambda_1, \dots \lambda_p$, then the set $\{\mathbf{v}_1, \dots \mathbf{v}_p\}$ is linearly independent.

- For a triangular matrix, eigenvalues are the diagonal entries:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \Rightarrow \qquad \begin{cases} \lambda_1 = a_{11} \\ \lambda_2 = a_{22} \\ \lambda_3 = a_{33} \end{cases}$$

(because its determinant is the product of its main diagonal)

**Example:** for $\mathbf{A} = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ find bases for its eigenspaces.

**Solution:**

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \left| \begin{bmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{bmatrix} \right|$$

$$= (4-\lambda)(1-\lambda)(8-\lambda) - 12 - 12 + 6(4-\lambda) + 2(8-\lambda) - 12(1-\lambda)$$

$$= -\lambda^3 + 13\lambda^2 - 40\lambda + 36$$

$$\cdots = \cdots$$

$$= (9-\lambda)(2-\lambda)(2-\lambda) = 0$$

so $\lambda_1 = 9$ and $\lambda_{2,3} = 2$.

**Example:** for $\mathbf{A} = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ find bases for its eigenspaces.

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...so $\lambda_1 = 9$ and $\lambda_{2,3} = 2$; then we solve $(\mathbf{A} - \lambda \mathbf{I})\,\mathbf{x} = \mathbf{0}$

$$\mathbf{A} - 9\mathbf{I} = \begin{bmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{A} - 2\mathbf{I} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x} = x_2 \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

So the bases are:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ for } \lambda_1 \qquad \text{and} \qquad \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ for } \lambda_{2,3}$$

Dilation e.g. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ we have $\lambda_{1,2} = 2$ and $\forall \mathbf{v} \in \mathbb{R}^2$

Shear e.g. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ we have $\lambda_{1,2} = 1$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Projection e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ we have $\lambda_1 = 1$ for $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

and $\lambda_2 = 0$ for $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Swap e.g. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ we have $\lambda_{1,2} = 1$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Rotation e.g. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ we have $\lambda_{1,2} = \pm \mathrm{i}$ and $\forall \mathbf{v} \in \mathbb{C}^2$

# Questions?



This week: quick test 5

(change of coordinates)

Next week: quick test 6

(linear transformations)