University of Technology Sydney

School of Mathematical and Physical Sciences

37233 Linear Algebra

Solutions 7

Question 1

(a) The domain of $\widehat{T}$ is $\mathbb{R}^3$ and the codomain is $\mathbb{R}^4$;

(b) The image of $\mathbf{u}$ turns out to be zero element of the codomain: $\mathbf{Au} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(c) To find the range of $\widehat{T}$, we check the columns of $\mathbf{T}$ for linear independence:

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

whereby row-reduction reveals two pivots, so the range of $\widehat{T}$ is a plane within $\mathbb{R}^4$, and we can select the first two vectors as its basis:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

(d) Given the row-reduced matrix above has one free variable, the general solution for the corresponding homogeneous system is a linear combination of only one vector. We already know such a vector from part (b), which can be reconfirmed from the reduced matrix, of course. For $\mathbf{Ax} = \mathbf{v}$, the particular solution is obvious, so then any vector of the form

$$\mathbf{x}_{\mathbf{v}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \qquad \forall x \in \mathbb{R}$$

produces $\mathbf{v}$ as the image.

However, an attempt to row-reduce the augmented matrix $[\mathbf{Aw}]$ reveals an inconsistency

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

therefore $\mathbf{w}$ is not in the range of $\mathbf{T}$ and cannot be obtained from any vector.

Question 2

(a) The domain and codomain are $\mathbb{R}^2$. The columns of $\mathbf{T}$ are linearly independent (not multiples of each other), so they span $\mathbb{R}^2$, and the range of $\widehat{T}$ is thus the entire $\mathbb{R}^2$.

(b) For the zero-corner of the square, it is obvious that the image is also zero because the transformation is linear; this also follows from multiplication. For the other corners:

$$\begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 3 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

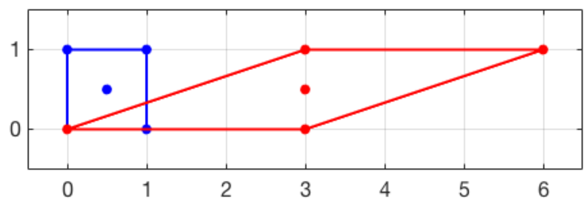
so the square undergoes a shear transformation with a 3-times horizontal stretch.

(c) To find the source vector $\mathbf{x}$ given its image $\mathbf{b}$ we need to solve the equation $\mathbf{Tx} = \mathbf{b}$. Row-reduction of the corresponding augmented matrix is straightforward:

$$\begin{bmatrix} 3 & 3 & 3 \\ 0 & 1 & 1/2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{bmatrix}$$

so $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$ is the source vector, which is the centre of the unit square.

Quite consistently, the image is the centre of the transformed square.



Question 3

(a) As the transformation runs from standard vectors, the standard matrix is directly given by the columns which are vectors $\mathbf{y}_i$, as shown below in (b).

(b) The image is obtained as $\mathbf{Tv} = \begin{bmatrix} 3 & -5 & -6 \\ 2 & 0 & 3 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{pmatrix} 7 \\ 5 \\ -2 \end{pmatrix}$.

(c) The reverse transformation matrix $\mathbf{T}^{-1}$ is obtained by row-reducing $[\mathbf{T}|\mathbf{I}] \rightarrow [\mathbf{I}|\mathbf{T}^{-1}]$:

$$\begin{bmatrix} 3 & -5 & -6 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ -1 & 3 & 5 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -5 & -6 & 1 & 0 & 0 \\ 0 & 10/3 & 7 & -2/3 & 1 & 0 \\ 0 & 0 & 1/5 & 3/5 & -2/5 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7/2 & -15/2 \\ 0 & 1 & 0 & -13/2 & 9/2 & -21/2 \\ 0 & 0 & 1 & 3 & -2 & 5 \end{bmatrix}$$

(d) The image is then obtained as $\mathbf{T}^{-1}\,[\mathbf{u}]_{\mathcal{Y}} = \begin{bmatrix} -9/2 & 7/2 & -15/2 \\ -13/2 & 9/2 & -21/2 \\ 3 & -2 & 5 \end{bmatrix}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix}$.

Question 4

(a) The characteristic equation

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda)\Big((3-\lambda)(9-\lambda)+9\Big) = (3-\lambda)(6-\lambda)(6-\lambda)$$

reveals $\lambda_1 = 3$ and $\lambda_{2,3} = 6$.

(b) For λ_1 the $(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \mathbf{0}$ equation has the matrix

$$\begin{bmatrix} 0 & -3 & 0 \\ 0 & 0 & -3 \\ 0 & 3 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \forall x_1 \in \mathbb{R}$$

For λ_2 the $(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x} = \mathbf{0}$ equation has the matrix

$$\begin{bmatrix} -3 & -3 & 0 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{x} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \forall x_3 \in \mathbb{R}$$

Thereby each of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

spans an eigenspace corresponding to λ_1 and λ_2, respectively.