Supplementary slides on linear transformations

Theorem: A linear transformation $\widehat{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a one-to-one transformation if and only if the equation $\widehat{T}(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.

Proof: Since $\widehat{T}$ is linear then $\widehat{T}(\mathbf{0}) = \mathbf{0}$. If $\widehat{T}$ is one-to-one, then the equation $\widehat{T}(\mathbf{x}) = \mathbf{0}$ has at most one solution, $\mathbf{x} = \mathbf{0}$.

Suppose $\widehat{T}$ is *not* one-to-one. Then $\exists\, \mathbf{b} \in \mathbb{R}^m$ which is the image of at least 2 different vectors $\mathbf{u} \neq \mathbf{v}$ in $\mathbb{R}^n$: $\widehat{T}(\mathbf{u}) = \mathbf{b}$, $\widehat{T}(\mathbf{v}) = \mathbf{b}$.

But since $\widehat{T}$ is linear, $\widehat{T}(\mathbf{u} - \mathbf{v}) = \widehat{T}(\mathbf{u}) - \widehat{T}(\mathbf{v}) = \mathbf{b} - \mathbf{b} = \mathbf{0}$.

However $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ and then the equation $\widehat{T}(\mathbf{x}) = \mathbf{0}$ is going to have more than one solution: $\mathbf{0}$ and $\mathbf{u} - \mathbf{v}$.

Therefore, the assumption that $\widehat{T}$ is not one-to-one is inconsistent with only having a trivial solution, so it is one-to-one then.

Theorem: Let $\widehat{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation with the standard matrix $\mathbf{T}$. Then:

(a) $\widehat{T}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of $\mathbf{T}$ span $\mathbb{R}^m$.

(b) $\widehat{T}$ is one-to-one if and only if the columns of $\mathbf{T}$ are linearly independent.

Proof:

(a) The columns of $\mathbf{T}$ span $\mathbb{R}^m$ if and only if $\forall\, \mathbf{b} \in \mathbf{R}^m$ the equation $\mathbf{Tx} = \mathbf{b}$ is consistent, so it has a solution for every $\mathbf{b}$. This is true only when $\widehat{T}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^m$.

(b) From the previous theorem, if $\widehat{T}$ is one-to-one, the equation $\mathbf{Tx} = \mathbf{0}$ has only the trivial solution. This can only happen if the columns of $\mathbf{T}$ are linearly independent.

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ be a basis for a linear space V. Then coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Proof: Consider two arbitrary elements in V,

$$\mathbf{u} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$
$$\mathbf{v} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n$$

Then $\mathbf{u} + \mathbf{v} = (c_1 + d_1)\mathbf{b}_1 + \ldots + (c_n + d_n)\mathbf{b}_n$ and

$$[\mathbf{u} + \mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = [\mathbf{u}]_{\mathcal{B}} + [\mathbf{v}]_{\mathcal{B}}$$

Thus coordinate mapping preserves addition.

Proof (continuation): If r is an arbitrary scalar, then

$$\begin{aligned} r\mathbf{u} &= r(c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n) = \\ &= (rc_1)\mathbf{b}_1 + \ldots + (rc_n)\mathbf{b}_n \end{aligned}$$

so

$$[r\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix} = r \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = r\,[\mathbf{u}]_{\mathcal{B}}$$

Thus coordinate mapping also preserves scalar multiplication.

Therefore, coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a linear mapping.

Since $\mathcal{B}$ is a basis, the elements $\mathbf{b}_1 \ldots \mathbf{b}_n$ are linearly independent, therefore the transformation is one-to-one.