University of Technology Sydney
School of Mathematical and Physical Sciences

37233 Linear Algebra

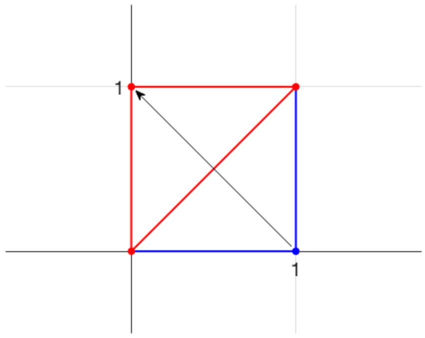
Tutorial 7 — solution guide

Question 1

For the zero-corner of the triangle, it is obvious that the image is also zero because the transformation is linear. For the other corners we have

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so the "horizontal" x and "vertical" y coordinates get swapped; in other words, the triangle is converted to its mirror image over the $x = y$ line.



Question 2

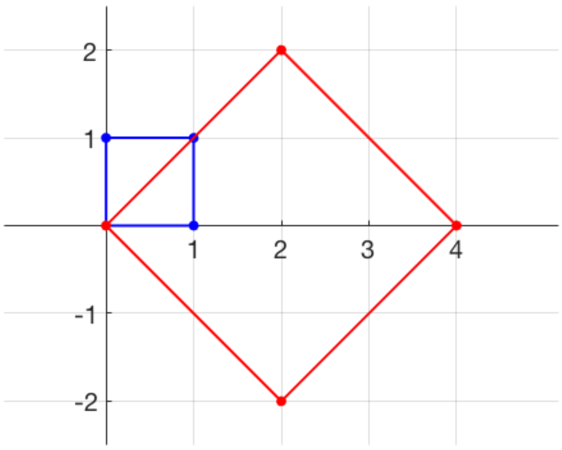
(a) The transformation is given by 2×2 matrix, so the domain and co-domain are $\mathbb{R}^2$. The columns are linearly independent, so there is a unique solution for any image, so the range is $\mathbb{R}^2$ and this is a one-to-one transformation.

(b) For the zero-corner of the square, it is obvious that the image is also zero because the transformation is linear. For the other corners we have

$$\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$



so the square is uniformly stretched $2\sqrt{2}$ times and rotated by $45°$ clockwise.

(c) By solving $\mathbf{Tx} = \mathbf{y}$, the unique solution to produce $\mathbf{y}$ is easily found as $\mathbf{x} = \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}$.

Question 3

(a) Checking actions of $\mathbf{A}$ over the given vectors yields

$$\mathbf{Av} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \cdot \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{v} \qquad \text{but} \qquad \mathbf{Au} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq c \cdot \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

thus $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ whereas $\mathbf{u}$ is not.

(b) If λ_1 is an eigenvalue of $\mathbf{A}$, then $\mathbf{Ax} = \lambda_1 \mathbf{x}$ for non-zero $\mathbf{x}$. Therefore we need to solve $\mathbf{Ax} - \lambda_1 \mathbf{x} = \mathbf{0}$, which is $(\mathbf{A} - \lambda_1 \mathbf{I})\,\mathbf{x} = \mathbf{0}$. Forming this matrix we get

$$\mathbf{A} - \lambda_1 \mathbf{I} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

and then row-reduction yields a set of non-trivial solutions:

$$\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{x} = c \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

thus $\lambda_1 = 7$ is an eigenvalue and the above $\mathbf{x}$ is the corresponding set of eigenvectors.

(c) Forming the characteristic equation we get

$$\det(\mathbf{A} - \lambda \mathbf{I}) = (1 - \lambda)(2 - \lambda) - 30 = \lambda^2 - 3\lambda - 28 = 0$$

where the solutions are $\lambda_1 = 7$ (as we already know, of course) and $\lambda_2 = -4$.

For the first root, the set of eigenvectors is already known, and for the second root

$$\mathbf{A} - \lambda_2 \mathbf{I} = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \qquad \Rightarrow \qquad \mathbf{y} = c \cdot \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

Question 4

(a) Yes, matrix $\mathbf{T} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ or $\mathbf{T} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) No, as can be easily concluded by seeing zero vector not mapped to zero vector.

(c) Yes, matrix $\mathbf{T} = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$ or $\mathbf{T} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$.

(d) Yes, matrix $\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ or $\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

(e) Yes, matrix $\mathbf{T} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ or $\mathbf{T} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$.