37233 Linear Algebra

Mid-term revision

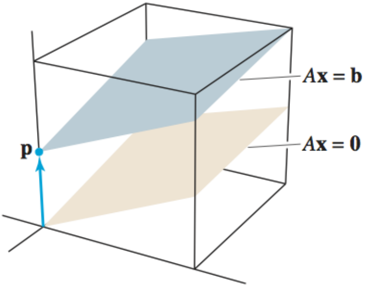
Key topics

- Linear systems, homogeneous and inhomogeneous
- Linear spaces and subspaces
- Span and linear (in)dependence
- Basis and dimension
- Coordinates and change of basis
- Linear transformations, eigenvalues, eigenvectors
- The invertible matrix theorem

Homogeneous and inhomogeneous systems

Suppose $\mathbf{Ax} = \mathbf{b}$ is consistent for some $\mathbf{b}$ and let $\mathbf{p}$ be a solution.

Then the solution set of $\mathbf{Ax} = \mathbf{b}$ is a set of all vectors in the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_0$, where $\mathbf{v}_0$ is any solution of the homogeneous equation $\mathbf{Ax} = \mathbf{0}$.



For $m \times n$ matrix $\mathbf{A}$, the following statements are equivalent:

- For each $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{Ax} = \mathbf{b}$ has a solution
- The columns of $\mathbf{A}$ span $\mathbb{R}^m$
- $\mathbf{A}$ has a pivot position in every row.

Linear spaces

A *linear space* is a non-empty set V of objects, on which two operations — addition and multiplication by scalars — are defined, subject to these axioms: $\forall\,\mathbf{u},\,\mathbf{v},\,\mathbf{w} \in V$ and $\forall\, c,\, d \in \mathbb{R}$

(i) $(\mathbf{u}+\mathbf{v}) \in V$

(ii) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(iii) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$

(iv) $\exists\,\mathbf{0} \quad : \quad \mathbf{u}+\mathbf{0} = \mathbf{u}$

(v) $\forall\,\mathbf{u}\ \ \exists\,(-\mathbf{u}) \quad : \quad \mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

(vi) $c\,\mathbf{u} \in V$

(vii) $1\cdot\mathbf{u} = \mathbf{u}$

(viii) $c\,(d\,\mathbf{u}) = (c\,d)\,\mathbf{u}$

(ix) $(c+d)\,\mathbf{u} = c\,\mathbf{u}+d\,\mathbf{u}$

(x) $c\,(\mathbf{u}+\mathbf{v}) = c\,\mathbf{u}+c\,\mathbf{v}$

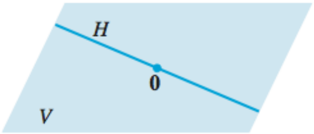
Properties: $0\cdot\mathbf{u} = \mathbf{0}$, $(-1)\cdot\mathbf{u} = -\mathbf{u}$, $c\cdot\mathbf{0} = \mathbf{0}$, etc.

Subspaces

Definition: A **subspace** H of a linear space V is a subset of elements with the following properties:

- Any element of H belongs to V
- H is closed under addition: $\forall\, (\mathbf{u}, \mathbf{v}) \in H, \; (\mathbf{u} + \mathbf{v}) \in H$
- H is closed under multiplication by scalars: $\forall\, \mathbf{u} \in H$ and $\forall\, c \in \mathbb{R}$, $c\,\mathbf{u} \in H$

Note: H necessarily includes the zero element of V



Every subspace is a linear space and satisfies the axioms.

Linear combinations and linear independence

Definition: $\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots c_n\mathbf{v}_n, \quad \mathbf{v}_i \in V, \quad c_i \in \mathbb{R}$
is called a *linear combination* of the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$
with coefficients $\{c_1, c_2 \ldots c_n\}$.

Definition: For elements $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p \in V$, the set of all their linear combinations is denoted by $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$ and is called the subset of V *spanned* (generated) by $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p$.

Definition: A set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_m\}$ is *linearly independent* if

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}$$

has only the trivial solution (with all $c_i = 0$).

- If a set contains $\mathbf{0}$, then this set is linearly dependent.
- Any set $\{\mathbf{v}_1, \ldots \mathbf{v}_p\} \in \mathbb{R}^n$ is linearly dependent if $p > n$.

Basis

Definition: An indexed set $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_p\}$ is a **basis** in V if:
$\mathcal{B}$ is a linearly independent set, and $V = \mathrm{Span}\{\mathbf{b}_1, \ldots \mathbf{b}_p\}$

- A basis is a smallest possible spanning set.
- A basis is a largest possible linearly independent set.

Definition: If V is spanned by a finite set, then V is called a finite-dimensional space, and the **dimension** of V, written as $\dim V = n$, is the number n of elements in a basis for V.

Warning: the dimension is a number of elements in a basis, **not** a number of components in each element.

The dimension of the $\{\mathbf{0}\}$ vector space is defined to be zero.

If V is not spanned by a finite set then V is *infinite-dimensional*.

Coordinate systems

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ be a basis for V, and $\mathbf{x} \in V$. The *coordinates* of $\mathbf{x}$ *relative to basis* $\mathcal{B}$ (or $\mathcal{B}$-coordinates of $\mathbf{x}$) are the coefficients $x_1, \ldots x_n$ such that

$$\mathbf{x} = x_1 \mathbf{b}_1 + \ldots + x_n \mathbf{b}_n$$

If $x_1, \ldots x_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, then the vector in $\mathbb{R}^n$

$$[\mathbf{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

provides the $\mathcal{B}$*-coordinates* of $\mathbf{x}$ (coordinates relative to $\mathcal{B}$).

Mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is unique for a given $\mathcal{B}$.

Change of basis

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots \mathbf{c}_n\}$ be bases of space V.

Then there is a unique $n \times n$ matrix $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} \, [\mathbf{x}]_{\mathcal{B}}$$

Columns of $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ are the $\mathcal{C}$-coordinates of the elements of $\mathcal{B}$.

That is, $\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \ldots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$

$\mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ is called the *change of coordinate matrix* from $\mathcal{B}$ to $\mathcal{C}$.

Calculating a change of basis matrix for n-dimensional space:

$$\left[\begin{array}{cccc|cccc} \mathbf{c}_1 & \mathbf{c}_2 & \ldots & \mathbf{c}_n & \mathbf{b}_1 & \mathbf{b}_2 & \ldots & \mathbf{b}_n \end{array}\right] \to \left[\begin{array}{c|c} \mathbf{I} & \mathbf{P}_{\mathcal{C}\leftarrow\mathcal{B}} \end{array}\right]$$

Linear transformations

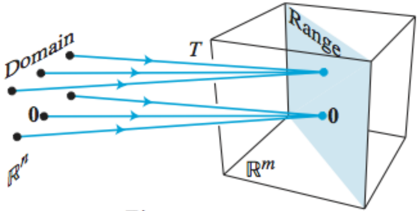
Definition: Transformation (mapping, functional) $\widehat{T}$ from $\mathbb{R}^n$ into $\mathbb{R}^m$ is a rule assigning a vector $\widehat{T}(\mathbf{x}) = \mathbf{y} \in \mathbb{R}^m$ to each vector $\mathbf{x} \in \mathbb{R}^n$

- $\mathbb{R}^n$ is called the **domain**, and $\mathbb{R}^m$ the **codomain** of $\widehat{T}$
- Vector $\mathbf{y} = \widehat{T}(\mathbf{x})$ is called the **image** of $\mathbf{x}$
- The set of all images $\left\{\widehat{T}(\mathbf{x})\right\}$ is called the **range** of $\widehat{T}$
- For $\mathbf{x} \in \mathbb{R}^n$, a linear transformation $\widehat{T}(\mathbf{x})$ into $\mathbb{R}^m$ can be computed as $\mathbf{Ax}$ where $\mathbf{A}$ is an $m \times n$ matrix

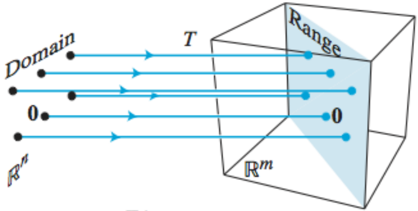
  If the domain of $\widehat{T}$ is $\mathbb{R}^n$ then $\mathbf{A}$ has n columns $\mathbf{a}_i$, and if the codomain of $\widehat{T}$ is $\mathbb{R}^m$ then $\mathbf{A}$ has m rows
- The range of $\widehat{T}$ is $\mathrm{Span}\{\mathbf{a}_1 \ldots \mathbf{a}_n\}$ and image of $\mathbf{x}$ is $\mathbf{Ax}$

Linear transformations

Definition: Transformation $\widehat{T} : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *one-to-one* if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at most one $\mathbf{x}$ in $\mathbb{R}^n$



T is *not* one-to-one



T is one-to-one

Generally in $\mathbb{R}^n$ $\quad \mathbf{x} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n$ with $\mathbf{e}_n \; : \; e_n^{(i)} = \delta_{ni}$

So $\widehat{T}(\mathbf{x}) = \mathbf{T}\mathbf{x}$ with standard matrix $\mathbf{T} = \left[\; \widehat{T}(\mathbf{e}_1) \;\middle|\; \ldots \;\middle|\; \widehat{T}(\mathbf{e}_n) \;\right]$

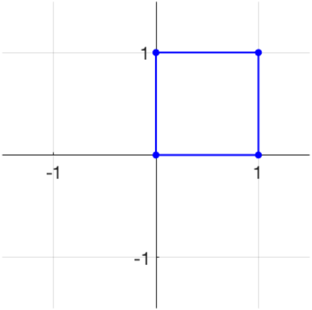
$\widehat{T}$ is one-to-one if and only if $\mathbf{T}$ columns are linearly independent.

Visualisation of linear transformations

A convenient way to visualise transformations in $\mathbb{R}^2$ is to depict their action on a unit square: a square made by vectors

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Plotting the resulting points and connecting these with lines (transformation is linear) shows how the shape is changed.

Eigenvectors and eigenvalues

Definition: An *eigenvector* of a square matrix $\mathbf{A}$ is a nonzero vector $\mathbf{v}$ such that $\mathbf{Av} = \lambda\mathbf{v}$ for some scalar λ.

A scalar λ is called an *eigenvalue* of $\mathbf{A}$ if there is a nontrivial solution $\mathbf{v}$ of $\mathbf{Av} = \lambda\mathbf{v}$.

Such an $\mathbf{v}$ is an *eigenvector corresponding to* λ.

As a *linear transformation* $\mathbf{A}$ performs *scaling* of its eigenvectors

λ is an eigenvalue for $\mathbf{A}$ if and only there is a nontrivial solution to

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

Therefore, λ can be found by solving a *characteristic equation*:

$$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$$

Note: there may be complex roots to the characteristic equation.

Eigenvectors and eigenvalues

- 0 is an eigenvalue of $\mathbf{A}$ if and only if $\mathbf{A}$ is singular.

- All the eigenvectors of $\mathbf{A}$ corresponding to a given eigenvalue, span the corresponding *eigenspace*

- If $\mathbf{v}_1, \dots \mathbf{v}_p$ are eigenvectors corresponding to different eigenvalues $\lambda_1, \dots \lambda_p$, then the set $\{\mathbf{v}_1, \dots \mathbf{v}_p\}$ is linearly independent.

- For a triangular matrix, eigenvalues are the diagonal entries:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \qquad \Rightarrow \qquad \begin{cases} \lambda_1 = a_{11} \\ \lambda_2 = a_{22} \\ \lambda_3 = a_{33} \end{cases}$$

The invertible matrix theorem (summary of results)

Statements equivalent to $\mathbf{A}$ being an $n \times n$ invertible matrix:

- There is an $n \times n$ matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$
- $\mathbf{A}$ has n pivot positions in the REF form
- $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution
- The columns (rows) of $\mathbf{A}$ form a linearly independent set
- The columns (rows) of $\mathbf{A}$ span $\mathbb{R}^n$
- The columns (rows) of $\mathbf{A}$ form a basis of $\mathbb{R}^n$
- $\widehat{T}: \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is one-to-one
- $\widehat{T}: \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
- $\mathrm{Col}\,\mathbf{A} = \mathrm{Row}\,\mathbf{A} = \mathbb{R}^n$
- $\mathrm{Nul}\,\mathbf{A} = \{\mathbf{0}\}$ and $\dim(\mathrm{Nul}\,\mathbf{A}) = 0$
- $\dim(\mathrm{Col}\,\mathbf{A}) = \dim(\mathrm{Row}\,\mathbf{A}) = n$
- $\mathrm{rank}\,\mathbf{A} = n$
- The eigenvalues of $\mathbf{A}$ are non-zero

Advice for solving test problems

- Identify and attack easier problems first
- If stuck, switch to another problem and retry later
- Keep to explicit radicals and rational fractions (such as $\dfrac{5}{\sqrt{3}}$, not "2.88675")
- If the numbers get really uncomfortable, something is probably wrong
- Articulate your solutions as much as possible (explain what you are doing)
- Check back the final answer where possible

Mid-term class test summary

- 4 problems (with sub-questions):
  10 marks per problem, 40 marks in total
- Topics 1, 3, 4, 5 and 7; as well as 2 for solving
  (as numbered in Canvas modules)
- 110 minutes time for completion