Matrix subspaces

- Subspaces of a matrix:
  - Column space
  - Null space
  - Row space
- Bases for matrix subspaces
- Dimensions of matrix subspaces
- Rank of a matrix and the rank theorem

Definition: A **subspace** H of a linear space V is a subset of elements with the following properties:

(i) H is closed under addition: $\forall\,(\mathbf{u},\mathbf{v}) \in H,\ \ (\mathbf{u}+\mathbf{v}) \in H$

(ii) H is closed under multiplication by scalars:
$\forall\,\mathbf{u} \in H$ and $\forall\,c \in \mathbb{R}$, $c\,\mathbf{u} \in H$

Definition: Set $\mathcal{B} = \{\mathbf{b}_1, \ldots\ \mathbf{b}_p\} \in V$ is a **basis** for V if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) $V = \mathrm{Span}\{\mathbf{b}_1, \ldots\ \mathbf{b}_p\}$

Definition: If V is spanned by a finite set, then V is called a *finite-dimensional* space, and the **dimension** of V, written as $\dim V = n$, is the number n of elements in a basis for V.

A specific set of vector subspaces arises
in relation to linear systems of equations.

For a given matrix $\mathbf{A}$, the following spaces are defined:

- Column space
- Null space
- Row space

Definition: The **column space** of an $m \times n$ matrix $\mathbf{A}$ is the set of all linear combinations of the columns of $\mathbf{A}$:

$$\operatorname{Col} \mathbf{A} = \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots \mathbf{a}_n\}$$

- Each $\mathbf{a}_i \in \mathbb{R}^m$, therefore $\operatorname{Col} \mathbf{A}$ is a subspace of $\mathbb{R}^m$

  (a spanning set is a subspace of the corresponding vector space)

- In terms of solving linear systems:

  $\operatorname{Col} \mathbf{A} = \{\mathbf{b}\}$ such that $\mathbf{b} = \mathbf{A}\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$

  (since $\mathbf{A}\mathbf{x}$ is a linear combination of the columns of $\mathbf{A}$)

- $\operatorname{Col} \mathbf{A} = \mathbb{R}^m$ if and only if $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution $\forall\, \mathbf{b} \in \mathbb{R}^m$

Example: Determine then if the following vectors are in $\operatorname{Col}\mathbf{A}$:

$$\mathbf{v} = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 0 \\ -7 \\ 7 \end{bmatrix}; \qquad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 6 & -1 \\ -7 & 0 \end{bmatrix}$$

Solution: Check if $\mathbf{Ax} = \mathbf{v}$, $\mathbf{Ax} = \mathbf{u}$ have a solution:

$$\mathbf{v}: \qquad \mathbf{A} = \left[\begin{array}{rr|r} 1 & 1 & 2 \\ 6 & -1 & 5 \\ -7 & 0 & 7 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{rr|r} 1 & 1 & 2 \\ 0 & -7 & -7 \\ 0 & 0 & 14 \end{array}\right]$$

so the system is inconsistent, thus $\mathbf{v} \notin \operatorname{Col}\mathbf{A}$

$$\mathbf{u}: \qquad \mathbf{A} = \left[\begin{array}{rr|r} 1 & 1 & 0 \\ 6 & -1 & -7 \\ -7 & 0 & 7 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{rr|r} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

so the system is consistent, thus $\mathbf{u} \in \operatorname{Col}\mathbf{A}$

Example: Find matrix $\mathbf{A}$ such that $W = \operatorname{Col}\mathbf{A}$ if

$$W = \left\{ \begin{bmatrix} a+b \\ 6a-b \\ -7a \end{bmatrix} \right\}, \quad a,\, b \in \mathbb{R}$$

Solution: Write W as a linear combination

$$W = \left\{ a \begin{bmatrix} 1 \\ 6 \\ -7 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

The vectors in the above spanning set are the columns of $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 6 & -1 \\ -7 & 0 \end{bmatrix} \qquad \text{and} \qquad W = \operatorname{Col}\mathbf{A}$$

Theorem:

- Pivot columns of $\mathbf{A}$ form a basis for $\operatorname{Col} \mathbf{A}$;
- $\dim(\operatorname{Col} \mathbf{A})$ is the number of pivot columns.

Proof: Let $\mathbf{B}$ be the REF form of $\mathbf{A}$. The pivot columns of $\mathbf{B}$ are linearly independent. Then the corresponding columns of $\mathbf{A}$ are also linearly independent (any linear relation between $\mathbf{a}_i$ implies the same linear relation for $\mathbf{b}_i$, via row-equivalence).

Thus, any non-pivot column of $\mathbf{A}$ is a linear combination of the pivot columns of $\mathbf{A}$. Then the non-pivot columns can be discarded from the spanning set for $\operatorname{Col} \mathbf{A}$, by the spanning set theorem. This leaves the pivot columns of $\mathbf{A}$ as a basis for $\operatorname{Col} \mathbf{A}$.

Note: It is important that the pivot columns of $\mathbf{A}$ itself, and not those of the REF form, are a basis for $\operatorname{Col} \mathbf{A}$. The column space of $\mathbf{B}$ is not necessarily the same as the column space of $\mathbf{A}$.

Example: Find a basis for $\mathrm{Col}\,\mathbf{A}$ of $\mathbf{A} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$.

We obtain with row reduction

$$\to \quad \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 0 & 0 & 2 & -2 & 13 \\ 0 & 0 & 1 & -1 & 8 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \mathbf{B}$$

The REF form shows that three columns (e.g. $\mathbf{b}_1$, $\mathbf{b}_3$ and $\mathbf{b}_5$) are linearly independent, whereas $\mathbf{b}_2 = 4\mathbf{b}_1$ and $\mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$.

Therefore we can retain $\mathbf{a}_1$, $\mathbf{a}_3$ and $\mathbf{a}_5$ while $\mathbf{a}_2 = 4\mathbf{a}_1$ and $\mathbf{a}_4 = 2\mathbf{a}_1 - \mathbf{a}_3$ can be discarded from the spanning set.

Thus $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ is a basis for $\mathrm{Col}\,\mathbf{A}$.

More explicitly, with that example we found the bases as:

$$\mathcal{B} = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \right\} \quad \text{for} \quad \operatorname{Col}\left(\begin{bmatrix}1 & 4 & 0 & 2 & 0\\0 & 0 & 1 & -1 & 0\\0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix} \right)$$

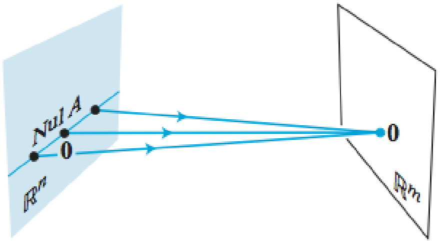
$$\mathcal{A} = \left\{ \begin{bmatrix}1\\3\\2\\5\end{bmatrix}, \begin{bmatrix}0\\1\\1\\2\end{bmatrix}, \begin{bmatrix}-1\\5\\2\\8\end{bmatrix} \right\} \quad \text{for} \quad \operatorname{Col}\left(\begin{bmatrix}1 & 4 & 0 & 2 & -1\\3 & 12 & 1 & 5 & 5\\2 & 8 & 1 & 3 & 2\\5 & 20 & 2 & 8 & 8\end{bmatrix} \right)$$

However even though matrices $\mathbf{A}$ and $\mathbf{B}$ are row equivalent, neither $\mathcal{A}$ is a basis for $\operatorname{Col}\mathbf{B}$, nor $\mathcal{B}$ is for $\operatorname{Col}\mathbf{A}$.

Only the indices of the columns which make the basis are the same.

Definition: The **null space** of an $m \times n$ matrix $\mathbf{A}$ is the set of all solutions to $\mathbf{Ax} = \mathbf{0}$:

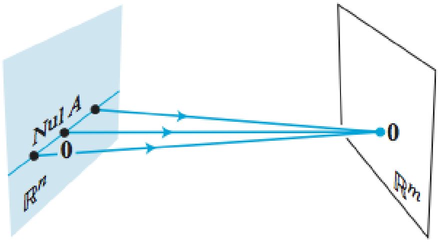
$$\mathrm{Nul}\,\mathbf{A} = \{\mathbf{x} : \ \mathbf{x} \in \mathbb{R}^n \ \text{ and } \ \mathbf{Ax} = \mathbf{0}\}$$



For an $m \times n$ matrix $\mathbf{A}$, $\operatorname{Nul}\mathbf{A}$ is a subspace of $\mathbb{R}^n$.

Proof: Consider $\{\mathbf{u},\, \mathbf{v}\} \in \mathbb{R}^n$ such that $\{\mathbf{u},\, \mathbf{v}\} \in \operatorname{Nul}\mathbf{A}$.
This implies that $\mathbf{A}\mathbf{u} = \mathbf{0}$ and $\mathbf{A}\mathbf{v} = \mathbf{0}$.
Then $\mathbf{A}(\mathbf{u}+\mathbf{v}) = \mathbf{A}\mathbf{u} + \mathbf{A}\mathbf{v} = \mathbf{0}+\mathbf{0} = \mathbf{0}$ so $(\mathbf{u}+\mathbf{v}) \in \operatorname{Nul}\mathbf{A}$.
Also $\mathbf{A}(c\,\mathbf{u}) = c(\mathbf{A}\mathbf{u}) = c \cdot \mathbf{0} = \mathbf{0}$, so $c\,\mathbf{u} \in \operatorname{Nul}\mathbf{A}$.

Therefore, $\operatorname{Nul}\mathbf{A}$ is a subspace of $\mathbb{R}^n$.



Consider the following system of homogeneous equations:

$$\begin{cases} x_1 - 3x_2 - 2x_3 = 0 \\ -5x_1 + 9x_2 + x_3 = 0 \end{cases} \quad \Leftrightarrow \quad \mathbf{A} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$$

Check if $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$ belong to $\operatorname{Nul}\mathbf{A}$:

$$\mathbf{Au} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{Aw} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

Thus $\mathbf{u} \in \operatorname{Nul}\mathbf{A}$ however $\mathbf{w} \notin \operatorname{Nul}\mathbf{A}$.

Example: Let H be the set of all vectors of the form $\mathbf{h} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

with the coordinates satisfying equations: $\begin{cases} b = c - a \\ d = a - 2b + 5c \end{cases}$

By rearranging the governing equations we obtain

$$\begin{cases} -a - b + c = 0 \\ a - 2b + 5c - d = 0 \end{cases} \quad \Leftrightarrow \quad \begin{bmatrix} -1 & -1 & 1 & 0 \\ 1 & -2 & 5 & -1 \end{bmatrix} \mathbf{h} = \mathbf{0}$$

Therefore H is the null space of the above matrix.
Given the earlier theorem, H is a subspace of $\mathbb{R}^4$.

Note: For a relation between the coordinates such that the corresponding system were *inhomogeneous*, the set would *not* be a subspace (as $\mathbf{0}$ is not a solution of an inhomogeneous system).

There is no immediate connection between $\mathrm{Nul}\,\mathbf{A}$ and a_{ij}.

Solving $\mathbf{Ax} = \mathbf{0}$ provides an explicit description of $\mathrm{Nul}\,\mathbf{A}$.

Example: Find a spanning set for the null space of

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

(1): find a general solution of $\mathbf{Ax} = \mathbf{0}$ in terms of free variables:

$$\to \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 13 & 26 & -26 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, x_1, x_3 are basic, and x_2, x_4, x_5 are free variables:

$x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = -2x_4 + 2x_5$.

(2): Using $x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = -2x_4 + 2x_5$
we decompose the general solution as a linear combination:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

where we can introduce

$$\mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_5 = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

and thereby $\operatorname{Nul}\mathbf{A} = \operatorname{Span}\{\mathbf{u}_2, \mathbf{u}_4, \mathbf{u}_5\}$.

Note: Each of $\mathbf{u}_2$, $\mathbf{u}_4$, $\mathbf{u}_5$ vectors corresponds to a free variable so it has an entry which is only given by that free variable, and cannot be obtained through the other free variables:

$$\mathbf{x} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ \mathbf{1} \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ \mathbf{1} \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ \mathbf{1} \end{bmatrix}$$

So all the three vectors are essential for the spanning set.

Thereby the spanning set for $\mathrm{Nul}\,\mathbf{A}$ is linearly independent, and so $x_2\mathbf{u}_2 + x_4\mathbf{u}_4 + x_5\mathbf{u}_5 = \mathbf{0}$ is only true if $x_2 = x_4 = x_5 = 0$.

Solving $\mathbf{Ax} = \mathbf{0}$ automatically produces a basis for $\mathrm{Nul}\,\mathbf{A}$.

Generally, when $\mathrm{Nul}\,\mathbf{A} \neq \mathbf{0}$, the number of linearly independent vectors in its spanning set equals to the number of free variables.

Thus the resulting spanning vectors form a basis for $\mathrm{Nul}\,\mathbf{A}$,

and $\dim(\mathrm{Nul}\,\mathbf{A})$ equals to the number of free variables.

Example: returning to the above example

$$\mathbf{x} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \equiv x_2\mathbf{u}_2 + x_4\mathbf{u}_4 + x_5\mathbf{u}_5$$

Vectors $\mathbf{u}_2$, $\mathbf{u}_4$, $\mathbf{u}_5$ span $\mathrm{Nul}\,\mathbf{A}$ and are linearly independent, therefore $\mathcal{U} = \{\mathbf{u}_2, \mathbf{u}_4, \mathbf{u}_5\}$ is a basis for $\mathrm{Nul}\left(\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}\right)$.

Example: Check if $\mathbf{u}$ and $\mathbf{v}$ belong to $\operatorname{Nul}\mathbf{A}$, $\operatorname{Col}\mathbf{A}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(a) Clearly, $\mathbf{u} \notin \operatorname{Col}\mathbf{A}$, because $\operatorname{Col}\mathbf{A}$ is a subspace of $\mathbb{R}^3$.

(b) To find out if $\mathbf{u} \in \operatorname{Nul}\mathbf{A}$, there is no need for an explicit description of $\operatorname{Nul}\mathbf{A}$; we just check if $\mathbf{A}\mathbf{u} = \mathbf{0}$:

$$\begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \mathbf{0} \qquad \Rightarrow \qquad \mathbf{u} \notin \operatorname{Nul}\mathbf{A}$$

Example: Check if $\mathbf{u}$ and $\mathbf{v}$ belong to $\mathrm{Nul}\,\mathbf{A}$, $\mathrm{Col}\,\mathbf{A}$ if

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(c) Clearly, $\mathbf{v} \notin \mathrm{Nul}\,\mathbf{A}$, because $\mathrm{Nul}\,\mathbf{A}$ is a subspace of $\mathbb{R}^4$.

(d) If $\mathbf{v} \in \mathrm{Col}\,\mathbf{A}$, it must be a solution to $\mathbf{A}\mathbf{x} = \mathbf{v}$, so we form the required augmented matrix and check with row reduction:

$$\left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ -2 & -5 & 7 & 3 & -1 \\ 3 & 7 & -8 & 6 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{array}\right]$$

We see that $\mathbf{A}\mathbf{x} = \mathbf{v}$ is consistent, so $\mathbf{v} \in \mathrm{Col}\,\mathbf{A}$.

Examples: Revisit a couple of earlier examples:

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns are 1 and 3, and free variables are x_2, x_4 and x_5.

Therefore $\dim(\mathrm{Nul}\,\mathbf{A}) = 3$ and $\dim(\mathrm{Col}\,\mathbf{A}) = 2$.

$$\mathbf{B} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns are 1, 3 and 5, and free variables are x_2 and x_4.

Therefore $\dim(\mathrm{Nul}\,\mathbf{B}) = 2$ and $\dim(\mathrm{Col}\,\mathbf{B}) = 3$.

Summary: for a given $m \times n$ matrix $\mathbf{A}$

- $\operatorname{Nul}\mathbf{A}$ is a subspace of $\mathbb{R}^n$;
  $\operatorname{Col}\mathbf{A}$ is a subspace of $\mathbb{R}^m$.
- $\operatorname{Nul}\mathbf{A}$ is implicitly defined by $\mathbf{A}\mathbf{x} = \mathbf{0}$;
  $\operatorname{Col}\mathbf{A}$ is explicitly defined as $\operatorname{Span}\{\mathbf{a}_i\}$.
- A vector of $\operatorname{Nul}\mathbf{A}$ is obtained by solving $\mathbf{A}\mathbf{x} = \mathbf{0}$;
  A vector of $\operatorname{Col}\mathbf{A}$ is obtained as a linear combination of $\{\mathbf{a}_i\}$.
- Checking if $\mathbf{v} \in \operatorname{Nul}\mathbf{A}$ is done by computing if $\mathbf{A}\mathbf{v} = \mathbf{0}$;
  Checking if $\mathbf{v} \in \operatorname{Col}\mathbf{A}$ requires solving $\mathbf{A}\mathbf{x} = \mathbf{v}$.
- $\dim(\operatorname{Nul}\mathbf{A})$ equals to the number of free variables;
  $\dim(\operatorname{Col}\mathbf{A})$ equals to the number of pivot columns.
- $\operatorname{Nul}\mathbf{A} = \{\mathbf{0}\}$ if and only if $\mathbf{A}\mathbf{x} = \mathbf{0}$ only for $\mathbf{x} = \mathbf{0}$;
  $\operatorname{Col}\mathbf{A} = \mathbb{R}^m$ if and only if $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution $\forall\, \mathbf{b} \in \mathbb{R}^m$.

If $\mathbf{A}$ is an $m \times n$ matrix, each row of $\mathbf{A}$ has n entries and thus can be identified with a vector in $\mathbb{R}^n$.

Definition:

The set of all linear combinations of row vectors of $\mathbf{A}$ is called the *row space* of $\mathbf{A}$, denoted as $\mathrm{Row}\,\mathbf{A}$.

Notes:

- Each row has n entries, so $\mathrm{Row}\,\mathbf{A}$ is a subspace of $\mathbb{R}^n$.
- Since the rows of $\mathbf{A}$ are the columns of $\mathbf{A}^\mathsf{T}$,

$$\mathrm{Row}\,\mathbf{A} = \mathrm{Col}\,\mathbf{A}^\mathsf{T} \qquad \text{as well as} \qquad \mathrm{Row}\,\mathbf{A}^\mathsf{T} = \mathrm{Col}\,\mathbf{A}$$

Theorem: If two matrices $\mathbf{A}$ and $\mathbf{B}$ are row-equivalent, then their row spaces are equal. If $\mathbf{B}$ is in echelon form, the non-zero rows of $\mathbf{B}$ form a basis for the row space of both $\mathbf{A}$ and $\mathbf{B}$.

Proof: If $\mathbf{B}$ is obtained from $\mathbf{A}$ by row operations, then every row of $\mathbf{B}$ is a linear combinations of the rows of $\mathbf{A}$.

Thereby, any linear combination of the rows of $\mathbf{B}$ is a linear combination of the rows of $\mathbf{A}$. Thus $\mathrm{Row}\,\mathbf{B} \subset \mathrm{Row}\,\mathbf{A}$.

Given that row operations are reversible, the same arguments lead to the conclusion that $\mathrm{Row}\,\mathbf{A} \subset \mathrm{Row}\,\mathbf{B}$.

Therefore, it must be that $\mathrm{Row}\,\mathbf{A} = \mathrm{Row}\,\mathbf{B}$.

Finally, if $\mathbf{B}$ is in echelon form, its non-zero rows are linearly independent. Thus the non-zero rows of $\mathbf{B}$ form a basis of the common row space of $\mathbf{B}$ and $\mathbf{A}$.

Example: Find bases and dimensions for $\mathrm{Col}\,\mathbf{A}$, $\mathrm{Nul}\,\mathbf{A}$, $\mathrm{Row}\,\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \rightarrow$$

$$\rightarrow \begin{bmatrix} 2 & 5 & -8 & 0 & 17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \mathbf{B}$$

Non-zero rows of $\mathbf{B}$ are the first three rows: $\dim(\mathrm{Row}\,\mathbf{A}) = 3$.

Pivot columns in $\mathbf{B}$ are columns 1, 2 and 4: $\dim(\mathrm{Col}\,\mathbf{A}) = 3$.

There are two free variables in $\mathbf{B}$: $\dim(\mathrm{Nul}\,\mathbf{A}) = 2$.

A basis for $\operatorname{Row}\mathbf{A}$ (*and* $\operatorname{Row}\mathbf{B}$) is given by non-zero rows of $\mathbf{B}$:

$$\left\{ \begin{bmatrix} 2 \\ 5 \\ -8 \\ 0 \\ 17 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4 \\ 20 \end{bmatrix} \right\}$$

A basis for $\operatorname{Col}\mathbf{A}$ (but *not* $\operatorname{Col}\mathbf{B}$) is given by pivot columns in $\mathbf{A}$:

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

(but a basis for $\mathbf{B}$ is given by the pivot columns of $\mathbf{B}$)

Continue to REF to solve $\mathbf{Ax} = \mathbf{0}$; free variables are x_3 and x_5:

$$\mathbf{B} = \begin{bmatrix} 2 & 5 & -8 & 0 & 17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \equiv \mathbf{C}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 & -x_5 \\ 2x_3 & -3x_5 \\ x_3 & \\ & 5x_5 \\ & x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}.$$

Therefore basis for $\operatorname{Nul} \mathbf{A}$ is: $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$

Definition: $\operatorname{rank}\mathbf{A} = \dim(\operatorname{Col}\mathbf{A})$

(the *rank* of matrix $\mathbf{A}$ is the dimension of the column space of $\mathbf{A}$)

Notes

- Since $\operatorname{Row}\mathbf{A} = \operatorname{Col}\mathbf{A}^{\mathsf{T}}$, then $\dim(\operatorname{Row}\mathbf{A}) = \operatorname{rank}\mathbf{A}^{\mathsf{T}}$.
- The dimension of $\operatorname{Nul}\mathbf{A}$ is sometimes called the *nullity* of $\mathbf{A}$.

Theorem (the rank theorem): For an $m \times n$ matrix $\mathbf{A}$:

(i) $\dim(\operatorname{Col}\mathbf{A}) = \dim(\operatorname{Row}\mathbf{A}) = \operatorname{rank}\mathbf{A}$

(ii) $\operatorname{rank}\mathbf{A} + \dim(\operatorname{Nul}\mathbf{A}) = n$

Proof: By definition, $\operatorname{rank}\mathbf{A} = \dim(\operatorname{Col}\mathbf{A})$, which equals to the number of basis vectors for $\operatorname{Col}\mathbf{A}$, which is the number of pivot columns in $\mathbf{A}$. Equivalently, $\operatorname{rank}\mathbf{A}$ is then the number of pivot columns in an echelon form $\mathbf{B}$ of $\mathbf{A}$.

Because $\mathbf{B}$ has a non-zero row for each pivot, and these rows form a basis for $\operatorname{Row}\mathbf{A}$, $\operatorname{rank}\mathbf{A}$ is also the dimension of $\operatorname{Row}\mathbf{A}$,

$$\dim(\operatorname{Row}\mathbf{A}) = \dim(\operatorname{Col}\mathbf{A})$$

The dimension of $\operatorname{Nul}\mathbf{A}$ is the number of columns of $\mathbf{A}$ which correspond to free variables, so which are *not* the pivot columns.

The total number of columns n is the sum of the number of pivot columns and the number of columns without pivots, and therefore

$$\operatorname{rank}\mathbf{A} + \dim(\operatorname{Nul}\mathbf{A}) = n$$

Revisit again these earlier examples:

$$\mathbf{A} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are two pivot columns and three free variables.

Therefore $\operatorname{rank} \mathbf{A} = \dim(\operatorname{Col} \mathbf{A}) = 2$ and $\dim(\operatorname{Nul} \mathbf{A}) = 3$, so $\operatorname{rank} \mathbf{A} + \dim(\operatorname{Nul} \mathbf{A}) = 5$, equals to the number of columns of $\mathbf{A}$.

$$\mathbf{B} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three pivot columns and two free variables.

Therefore $\operatorname{rank} \mathbf{B} = \dim(\operatorname{Col} \mathbf{B}) = 3$ and $\dim(\operatorname{Nul} \mathbf{B}) = 2$, so $\operatorname{rank} \mathbf{B} + \dim(\operatorname{Nul} \mathbf{B}) = 5$, equals to the number of columns of $\mathbf{B}$.

Further examples:

$$\mathbf{A} = \begin{bmatrix} -3 & -1 & 1 \\ 1 & 2 & 3 \\ 2 & 5 & 9 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

There are three pivot columns and no free variables.

Therefore $\operatorname{rank}\mathbf{A} = \dim(\operatorname{Col}\mathbf{A}) = 3$ and $\dim(\operatorname{Nul}\mathbf{A}) = 0$, so $\operatorname{rank}\mathbf{A} + \dim(\operatorname{Nul}\mathbf{A}) = 3$, equals to the number of columns of $\mathbf{A}$.

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 5 \\ 2 & 1 & 3 \\ 5 & 2 & 8 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

There are two pivot columns and one free variable.

Therefore $\operatorname{rank}\mathbf{B} = \dim(\operatorname{Col}\mathbf{B}) = 2$ and $\dim(\operatorname{Nul}\mathbf{B}) = 1$, so $\operatorname{rank}\mathbf{B} + \dim(\operatorname{Nul}\mathbf{B}) = 3$, equals to the number of columns of $\mathbf{B}$.

Examples:

(a) If $\mathbf{A}$ is a 7×9 matrix with a two-dimensional null space, what is the rank of $\mathbf{A}$?

— As $\mathbf{A}$ has 9 columns, $\mathrm{rank}\,\mathbf{A} + 2 = 9$ and hence $\mathrm{rank}\,\mathbf{A} = 7$.

(b) Could a 6×9 matrix $\mathbf{B}$ have a two-dimensional null space?

— If a 6×9 matrix had a two-dimensional null space it would have to have rank 7 by the rank theorem.

However the columns of $\mathbf{B}$ are vectors in $\mathbb{R}^6$, and so $\dim(\mathrm{Col}\,\mathbf{B}) \leqslant 6$; therefore $\mathrm{rank}\,\mathbf{B} \leqslant 6$.

Thus $\mathbf{B}$ cannot have a two-dimensional null space.

Example:

Suppose the solution set of a homogeneous system of 40 equations in 42 variables is based on two solution vectors (so that any other solution is a linear combination of those two vectors).

The two vectors are linearly independent and span $\operatorname{Nul} \mathbf{A}$.

Therefore $\dim(\operatorname{Nul} \mathbf{A}) = 2$.

The matrix of the system $\mathbf{A}$ is a 40×42 matrix, so $n = 42$.

By the rank theorem, $\dim(\operatorname{Col} \mathbf{A}) = 42 - 2 = 40$.

Since $\mathbb{R}^{40}$ is the only subspace of $\mathbb{R}^{40}$ with dimension of 40, $\operatorname{Col} \mathbf{A}$ must be all of $\mathbb{R}^{40}$.

Then the inhomogeneous system $\mathbf{Ax} = \mathbf{b}$ has a solution $\forall \, \mathbf{b}$.

The invertible matrix theorem (summary of results)

Statements equivalent to $\mathbf{A}$ being an $n \times n$ invertible matrix:

- There is an $n \times n$ matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$
- $\mathbf{A}$ has n pivot positions in the REF form
- $\mathbf{A}\mathbf{x} = \mathbf{0}$ has only the trivial solution
- The columns (rows) of $\mathbf{A}$ form a linearly independent set
- The columns (rows) of $\mathbf{A}$ span $\mathbb{R}^n$
- The columns (rows) of $\mathbf{A}$ form a basis of $\mathbb{R}^n$
- $\widehat{T}:\ \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is one-to-one
- $\widehat{T}:\ \mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
- $\mathrm{Col}\,\mathbf{A} = \mathrm{Row}\,\mathbf{A} = \mathbb{R}^n$
- $\mathrm{Nul}\,\mathbf{A} = \{\mathbf{0}\}$ and $\dim(\mathrm{Nul}\,\mathbf{A}) = 0$
- $\dim(\mathrm{Col}\,\mathbf{A}) = \dim(\mathrm{Row}\,\mathbf{A}) = n$
- $\mathrm{rank}\,\mathbf{A} = n$

- Column space, row space and null space of a matrix
- Basis for column, row and null spaces
- Dimensions of column, row and null spaces
- Rank of a matrix: $\operatorname{rank} \mathbf{A} = \dim(\operatorname{Col} \mathbf{A})$
- Rank theorem: $\operatorname{rank} \mathbf{A} + \dim(\operatorname{Nul} \mathbf{A}) = n$
  (for an $m \times n$ matrix $\mathbf{A}$)

Questions?

Quick test 6 this week
(linear transformations)

No classes next week (StuVac)

Major Class Test (40 marks)
on Friday 26 April