University of Technology Sydney

School of Mathematical and Physical Sciences

37233 Linear Algebra

Solutions 8

Question 1

Row-reduction for $\mathbf{A}$ yields 3 pivots and one free variable:

$$\to \begin{bmatrix} 1 & 7 & -6 & -1 \\ 0 & -15 & 15 & -2 \\ 0 & 0 & 0 & -5/3 \\ 0 & 0 & 0 & -5/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so for homogeneous system $\mathbf{Az} = \mathbf{0}$ there is a non-trivial solution, and inhomogeneous system $\mathbf{Az} = \mathbf{b}$ does not have a solution $\forall\, \mathbf{b}$.

(a) There are three linearly independent columns in $\mathbf{A}$, for example, first, second and fourth. These columns span a subspace of $\mathbb{R}^4$:

$$\operatorname{Col}\mathbf{A} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ -8 \\ -9 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 1 \\ 3 \end{bmatrix} \right\}$$

Alternatively, any other two out of the first three columns can be used, along with, invariably, the fourth column.

Note that to write $\operatorname{Col}\mathbf{A}$, you can use all the columns; this does not change the span.

For the row space, the three non-zero rows of the REF form can be used:

$$\operatorname{Row}\mathbf{A} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

as well as the first three rows of the original matrix $\mathbf{A}$.

Note that to write $\operatorname{Row}\mathbf{A}$, you can still use all the rows; this does not change the span.

The null space is defined by the solution set for the homogeneous system:

$$\operatorname{Nul}\mathbf{A} = \operatorname{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(b) Bases for $\operatorname{Col}\mathbf{A}$, for $\operatorname{Nul}\mathbf{A}$, and for $\operatorname{Row}\mathbf{A}$ are shown above in (a).

(c) $\dim(\operatorname{Col}\mathbf{A}) = 3, \quad \dim(\operatorname{Nul}\mathbf{A}) = 1, \quad \dim(\operatorname{Row}\mathbf{A}) = 3.$

(d) The easiest check is for null space:

$$\mathbf{Ax} = \begin{bmatrix} 13 \\ -15 \\ -15 \\ 13 \end{bmatrix} \qquad \mathbf{Ay} = \begin{bmatrix} 40 \\ -100 \\ -50 \\ 90 \end{bmatrix} \qquad \mathbf{Az} = \begin{bmatrix} -42 \\ 48 \\ 54 \\ -36 \end{bmatrix}$$

so $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \notin \operatorname{Nul}\mathbf{A}$. The same conclusion can be reached by noticing that none of these vectors is not a multiple of the spanning vector for $\operatorname{Nul}\mathbf{A}$ listed in (a).

To check if $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \operatorname{Col}\mathbf{A}$, we row-reduce the corresponding triple-augmented matrix (so that you do the same operations once and for all — however keep in mind that the row-reduction algorithm may not "cross" the augmentation line):

$$\left[\begin{array}{rrrr|rrr} 1 & 7 & -6 & -1 & 1 & 0 & -2 \\ 1 & -8 & 9 & -3 & -1 & 2 & -4 \\ -2 & -9 & 7 & 1 & -3 & -6 & 2 \\ -2 & 6 & -8 & 3 & -1 & 10 & 0 \end{array}\right]$$

$$\cdots \to \left[\begin{array}{rrrr|rrr} 1 & 7 & -6 & -1 & 1 & 0 & -2 \\ 0 & -15 & 15 & -2 & -2 & 2 & -2 \\ 0 & 0 & 0 & -5/3 & -5/3 & 38/3 & -20/3 \\ 0 & 0 & 0 & -5/3 & -5/3 & -16/3 & -8/3 \end{array}\right]$$

$$\cdots \to \left[\begin{array}{rrrr|rrr} 1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

which only shows a consistent system for the first augmented column, so $\mathbf{x} \in \operatorname{Col}\mathbf{A}$, but $\mathbf{y}$, $\mathbf{z} \notin \operatorname{Col}\mathbf{A}$ (inconsistent system shown by second and third augmented columns).

Similarly, to check if $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \operatorname{Row}\mathbf{A}$, we row-reduce the corresponding triple-augmented transposed matrix $\mathbf{A}^\mathsf{T}$; the result is

$$\left[\begin{array}{rrrr|rrr} 1 & 1 & -2 & -2 & 1 & 0 & -2 \\ 7 & -8 & -9 & 6 & -1 & 2 & -4 \\ -6 & 9 & 7 & -8 & -3 & -6 & 2 \\ -1 & -3 & 1 & 3 & -1 & 10 & 0 \end{array}\right]$$

$$\cdots \to \left[\begin{array}{rrrr|rrr} 1 & 1 & -2 & -2 & 1 & 0 & -2 \\ 0 & -15 & 5 & 20 & -8 & 2 & 10 \\ 0 & 0 & -5/3 & -5/3 & 16/15 & 146/15 & -10/3 \\ 0 & 0 & 0 & 0 & -5 & -4 & 0 \end{array}\right]$$

$$\cdots \to \left[\begin{array}{rrrr|rrr} 1 & 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

so inconsistency for $\mathbf{x}$ and $\mathbf{y}$ but a solution for $\mathbf{z}$, thus $\mathbf{z} \in \operatorname{Row}\mathbf{A}$, but $\mathbf{x}$, $\mathbf{y} \notin \operatorname{Row}\mathbf{A}$.

Question 2

As it follows from the rank theorem, $\forall\,\mathbf{M}$ (for all cases): $\dim(\operatorname{Col}\mathbf{M})+\dim(\operatorname{Nul}\mathbf{M}) = 2$.

(a) Infinitely many possibilities, with any symmetric matrix, for example $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$.

For such a matrix, either $\dim(\operatorname{Col}\mathbf{A}) = \dim(\operatorname{Row}\mathbf{A}) = 2$ and $\dim(\operatorname{Nul}\mathbf{A}) = 0$, or $\dim(\operatorname{Col}\mathbf{A}) = \dim(\operatorname{Row}\mathbf{A}) = \dim(\operatorname{Nul}\mathbf{A}) = 1$, or, for the matrix of all zeros, $\dim(\operatorname{Col}\mathbf{A}) = \dim(\operatorname{Row}\mathbf{A}) = 0$ and $\dim(\operatorname{Nul}\mathbf{A}) = 2$.

(b) Infinitely many possibilities, for example $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

For all suitable cases, $\dim(\operatorname{Col}\mathbf{B}) = \dim(\operatorname{Nul}\mathbf{B}) = \dim(\operatorname{Row}\mathbf{B}) = 1$.

(c) Impossible. Suppose $\mathbf{x}$ belongs to $\operatorname{Nul}\mathbf{C}$ and $\operatorname{Row}\mathbf{C}$, then $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$, so $x_1^2 + x_2^2 = 0$ which is only possible if $x_1 = x_2 = 0$. Thus $\mathbf{0}$ is the only vector that belongs to $\operatorname{Nul}\mathbf{C}$ and $\operatorname{Row}\mathbf{C}$ at once. To have $\operatorname{Nul}\mathbf{C} = \operatorname{Row}\mathbf{C}$, this implies $\operatorname{Nul}\mathbf{C} = \operatorname{Row}\mathbf{C} = \operatorname{Span}\{\mathbf{0}\}$. However for a matrix with only zero rows, any vector belongs to $\operatorname{Nul}\mathbf{C}$, which leads to a contradiction. Therefore $\operatorname{Row}\mathbf{C} \neq \operatorname{Nul}\mathbf{C}$.

(d) Any matrix.

(e) This is only possible if $\dim(\operatorname{Col}\mathbf{E}) = \dim(\operatorname{Nul}\mathbf{E}) = 1$, which is the case for any non-zero matrix with linearly dependent columns, for example $\mathbf{E} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.
Clearly, $\dim(\operatorname{Row}\mathbf{E}) = 1$ in all such cases.

(f) Likewise, this is only possible if $\dim(\operatorname{Row}\mathbf{F}) = \dim(\operatorname{Nul}\mathbf{F}) = 1$, which is the case for any non-zero matrix with linearly dependent rows, for example $\mathbf{F} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.
Clearly, $\dim(\operatorname{Col}\mathbf{F}) = 1$ in all such cases.

(g) This is not possible, as the only possibility would be $\dim(\operatorname{Col}\mathbf{G}) = \dim(\operatorname{Row}\mathbf{G}) = 0$ and $\dim(\operatorname{Nul}\mathbf{G}) = 1$, however this combination does not satisfy the rank theorem.

(h) Given the rank theorem, this can be only satisfied if $\dim(\operatorname{Col}\mathbf{H}) = \dim(\operatorname{Row}\mathbf{H}) = 0$ and $\dim(\operatorname{Nul}\mathbf{H}) = 2$, which is the case only for the matrix of all zeros.

(i) The only possibility is $\dim(\operatorname{Col}\mathbf{J}) = \dim(\operatorname{Nul}\mathbf{J}) = \dim(\operatorname{Row}\mathbf{J}) = 1$, which is the case for any non-zero matrix with linearly dependent columns (and rows).

(j) This is only possible when $\dim(\operatorname{Col}\mathbf{K}) = \dim(\operatorname{Row}\mathbf{K}) = 2$ and $\dim(\operatorname{Nul}\mathbf{K}) = 0$, which is the case for any matrix with linearly independent columns.