University of Technology Sydney
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37233 Linear Algebra

Tutorial 8 — solution guide

Question 1

Further reduction towards REF yields $\mathbf{A} \rightarrow \begin{bmatrix} 1 & 0 & 15 & 0 & 7 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

(a)&(c)

$$\operatorname{Col}\mathbf{A} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ 10 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 5 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\operatorname{Row}\mathbf{A} = \left\{ \begin{bmatrix} 1 \\ -3 \\ 9 \\ 0 \\ -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ -4 \\ -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 10 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 15 \\ 0 \\ 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$\operatorname{Nul}\mathbf{A} = \left\{ \begin{bmatrix} -15 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$$

(b) $\dim(\operatorname{Nul}\mathbf{A}) = 2$, $\dim(\operatorname{Col}\mathbf{A}) = 3$ and $\dim(\operatorname{Row}\mathbf{A}) = 3$.

(c) Bases for $\operatorname{Nul}\mathbf{A}$, $\operatorname{Row}\mathbf{A}$, and, with grey colour, for $\operatorname{Col}\mathbf{A}$, are shown in (a).

Question 2

There are 9 equations so possible vectors of the right-hand side $\mathbf{b} \in \mathbb{R}^9$ and since there is a solution $\forall\,\mathbf{b}$, for the corresponding 9×10 matrix $\mathbf{A}$ we can state that $\operatorname{Col}\mathbf{A} = \mathbb{R}^9$ so $\dim(\operatorname{Col}\mathbf{A}) = 9$. Applying the rank theorem with $n = 10$, we see

$$\dim(\operatorname{Nul}\mathbf{A}) = 10 - 9 = 1.$$

Therefore, it is not possible to find two linearly independent solutions to the homogeneous system of $\mathbf{A}$. There is only one linearly independent vector in $\operatorname{Nul}\mathbf{A}$.

Question 3

(1) Note that cases (a) and (c) immediately imply that the size of $\mathbf{A}$ can only be 3×3.

(a) For example: $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 7 & 6 \end{bmatrix}$, with $\operatorname{rank} \mathbf{A} = 3$

(b) There is no such matrix. If $\mathbf{v}$ were in $\operatorname{Nul} \mathbf{A}$ and $\operatorname{Row} \mathbf{A}$, then each row of $\mathbf{A}$, and any linear combination of the rows, should give zero when multiplied by $\mathbf{v}$.
In particular, it should be $0 = \mathbf{v}^{\intercal}\mathbf{v} = 1^2 + 2^2 + 3^2$ which is certainly not true.

(c) If $\mathbf{v} \in \operatorname{Nul} \mathbf{A}$, then $\dim \operatorname{Nul} \mathbf{A} \geqslant 1$, then $\operatorname{rank} \mathbf{A} = 3 - \dim \operatorname{Nul} \mathbf{A} \leqslant 2$.

An example of satisfying matrix of rank 2 is $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \\ 3 & 0 & -1 \end{bmatrix}$

(2) All these facts can be established without knowing the explicit form of $\mathbf{B}$:

(d) Given that $\mathbf{v} \notin \operatorname{Col} \mathbf{B}$, the columns of $\mathbf{B}$ must be linearly dependent, so $\operatorname{rank} \mathbf{B} < 3$. At the same time, given that $\mathbf{v} \notin \operatorname{Nul} \mathbf{B}$, the matrix is non-zero, so then $\operatorname{rank} \mathbf{B} > 0$. Thus the rank of $\mathbf{B}$ is either 1 or 2.

(e) As the columns of $\mathbf{B}$ are linearly dependent, $\det \mathbf{B} = 0$.

Additional questions

Question 4

Reductions towards REF yield

$$\mathbf{A} \to \begin{bmatrix} 1 & -2 & 0 & 9 & -16 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{A}^{\intercal} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a)

$$\operatorname{Row} \mathbf{A} = \left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 9 \\ -16 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 5 \end{bmatrix} \right\} \qquad \text{and} \qquad \dim(\operatorname{Row} \mathbf{A}) = 2$$

(b)

$$\operatorname{Col} \mathbf{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\} \qquad \text{and} \qquad \dim(\operatorname{Col} \mathbf{A}) = 2$$

(c)

$$\operatorname{Nul}\mathbf{A} = \left\{ \begin{bmatrix} 2\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} -9\\0\\3\\1\\0 \end{bmatrix}, \begin{bmatrix} 16\\0\\-5\\0\\1 \end{bmatrix} \right\} \qquad \text{and} \qquad \dim(\operatorname{Nul}\mathbf{A}) = 3$$

(d)

$$\operatorname{Nul}\mathbf{A}^{\mathsf{T}} = \left\{ \begin{bmatrix} -5\\1\\1 \end{bmatrix} \right\} \qquad \text{and} \qquad \dim\left(\operatorname{Nul}\mathbf{A}^{\mathsf{T}}\right) = 1$$

$$\operatorname{Col}\mathbf{A}^{\mathsf{T}} = \left\{ \begin{bmatrix} 1\\-2\\3\\0\\-1 \end{bmatrix} \begin{bmatrix} 2\\-4\\7\\-3\\3 \end{bmatrix} \right\} \qquad \text{and} \qquad \dim\left(\operatorname{Col}\mathbf{A}^{\mathsf{T}}\right) = 2$$

$$\operatorname{Row}\mathbf{A}^{\mathsf{T}} = \left\{ \begin{bmatrix} 1\\0\\5 \end{bmatrix}, \begin{bmatrix} 0\\1\\-1 \end{bmatrix} \right\} \qquad \text{and} \qquad \dim\left(\operatorname{Row}\mathbf{A}^{\mathsf{T}}\right) = 2$$

Notes: $\operatorname{Nul}\mathbf{A}^{\mathsf{T}}$ is essentially different and unrelated to $\operatorname{Nul}\mathbf{A}$, whereas $\operatorname{Col}\mathbf{A}^{\mathsf{T}} = \operatorname{Row}\mathbf{A}$ and $\operatorname{Row}\mathbf{A}^{\mathsf{T}} = \operatorname{Col}\mathbf{A}$. The bases written above, being deduced from the REF form of $\mathbf{A}^{\mathsf{T}}$, look different but in fact they describe the same subspaces as the corresponding bases obtained in (a) and (b), as can be checked with row reduction.

Question 5

For the corresponding 12×8 matrix, it is therefore known that $\dim(\operatorname{Nul}\mathbf{A}) = 2$. Applying the rank theorem with $n = 8$, we find

$$\dim(\operatorname{Col}\mathbf{A}) = 8 - 2 = 6, \qquad \text{so} \qquad \operatorname{rank}\mathbf{A} = 6.$$

Hence the complete solution set for the system is covered by just 6 out of the 12 equations (and row reduction of $\mathbf{A}$ would result in 6 rows of only zeros out of 12).

Question 6

Reduction towards REF yields $\mathbf{A} \to \begin{bmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

$$\operatorname{Nul}\mathbf{A} = \left\{ \begin{bmatrix} -2\\1\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} -2\\0\\-3\\1\\0 \end{bmatrix}, \begin{bmatrix} -1\\0\\-1\\0\\1 \end{bmatrix} \right\}, \qquad \operatorname{Col}\mathbf{A} = \left\{ \begin{bmatrix} 1\\-1\\1 \end{bmatrix}, \begin{bmatrix} 0\\1\\-3 \end{bmatrix} \right\}$$

whereby it is found that $\dim(\operatorname{Nul}\mathbf{A}) = 3$ and $\dim(\operatorname{Col}\mathbf{A}) = 2$.

Question 7

Reductions (no row swaps) towards REF yield

$$\mathbf{A} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{A}^\mathsf{T} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a)

$$\operatorname{Row}\mathbf{A} = \operatorname{Col}\mathbf{A}^\mathsf{T} = \left\{ \begin{bmatrix} 3 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 5 \\ 3 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b)

$$\operatorname{Col}\mathbf{A} = \operatorname{Row}\mathbf{A}^\mathsf{T} = \left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(a-b)

$$\dim(\operatorname{Row}\mathbf{A}) = \dim(\operatorname{Col}\mathbf{A}) = \dim\left(\operatorname{Row}\mathbf{A}^\mathsf{T}\right) = \dim\left(\operatorname{Col}\mathbf{A}^\mathsf{T}\right) = 3$$

(c)

$$\operatorname{Nul}\mathbf{A} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \qquad \text{and} \quad \dim(\operatorname{Nul}\mathbf{A}) = 1$$

(d)

$$\operatorname{Nul}\mathbf{A}^\mathsf{T} = \{\mathbf{0}\} \qquad \text{and} \quad \dim\left(\operatorname{Nul}\mathbf{A}^\mathsf{T}\right) = 0$$

(thus, the homogeneous system of $\mathbf{A}^\mathsf{T}$ has only a trivial solution).